CE100 Algorithms and Programming II

Week-9 (Huffman Coding)

Spring Semester, 2021-2022

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Huffman Coding

Outline

- Heap Data Structure (Review Week-4)
- Heap Sort (Review Week-4)
- Huffman Coding

Huffman Codes

Huffman Codes for Compression

- Widely used and very effective for data compression
- Savings of 20% - 90% typical
  - (depending on the characteristics of the data)
- **In summary:** Huffman's greedy algorithm uses a **table of frequencies** of character occurrences to build up an optimal way of **representing each character as a binary string**.

Binary String Representation - Example

- Consider a data file with:
  - 100K characters
  - Each character is one of $\{a, b, c, d, e, f\}$
- Frequency of each character in the file:
  - **frequency:** $\overbrace{a}^{45K}, \overbrace{b}^{13K}, \overbrace{c}^{12K}, \overbrace{d}^{16K}, \overbrace{e}^{9K}, \overbrace{f}^{5K}$
- **Binary character code:** Each character is represented by a unique binary string.
- **Intuition:**
  - Frequent characters $\Leftrightarrow$ shorter codewords
  - Infrequent characters $\Leftrightarrow$ longer codewords

Binary String Representation - Example

characters	a	b	c	d	e	f
frequency	$45K$	$13K$	$12K$	$16K$	$9K$	$5K$
fixed-length	000	001	010	011	100	101
variable-length(1)	0	101	100	111	1101	1100
variable-length(2)	0	10	110	1110	11110	11111

- How many total bits needed for **fixed-length** codewords?
  $100K \times 3 = 300K\ bits$
- How many total bits needed for **variable-length(1)** codewords?
  $45K \times 1 + 13K \times 3 + 12K \times 3 + 16K \times 3 + 9K \times 4 + 5K \times 4 = 224K$
- How many total bits needed for **variable-length(2)** codewords?
  $45K \times 1 + 13K \times 2 + 12K \times 3 + 16K \times 4 + 9K \times 5 + 5K \times 5 = 241K$

Prefix Codes

- **Prefix codes:** No codeword is also a prefix of some other codeword
- **Example:**

characters	a	b	c	d	e	f
codeword	0	101	100	111	1101	1100

- It can be shown that:
  - Optimal data compression is achievable with a **prefix code**
- In other words, optimality is not lost due to **prefix-code** restriction.

Prefix Codes: Encoding

$$\begin{array}{ccccccc} \text{characters} & a & b & c & d & e & f \\ \text{codeword} & 0 & 101 & 100 & 111 & 1101 & 1100 \end{array}$$

- **Encoding:** Concatenate the codewords representing each character of the file
- **Example:** Encode file "abc" using the codewords above
  - $abc \Rightarrow 0.101.100 \Rightarrow 0101100$
- **Note:** "." denotes the concatenation operation. It is just for illustration purposes, and does not exist in the encoded string.

Prefix Codes: Decoding

- Decoding is quite simple with a prefix code
- The first codeword in an encoded file is unambiguous
  - *because no codeword is a prefix of any other*
- **Decoding algorithm:**
  - Identify the initial codeword
  - Translate it back to the original character
  - Remove it from the encoded file
  - Repeat the decoding process on the remainder of the encoded file.

Prefix Codes: Decoding - Example

characters	a	b	c	d	e	f
codeword	0	101	100	111	1101	1100

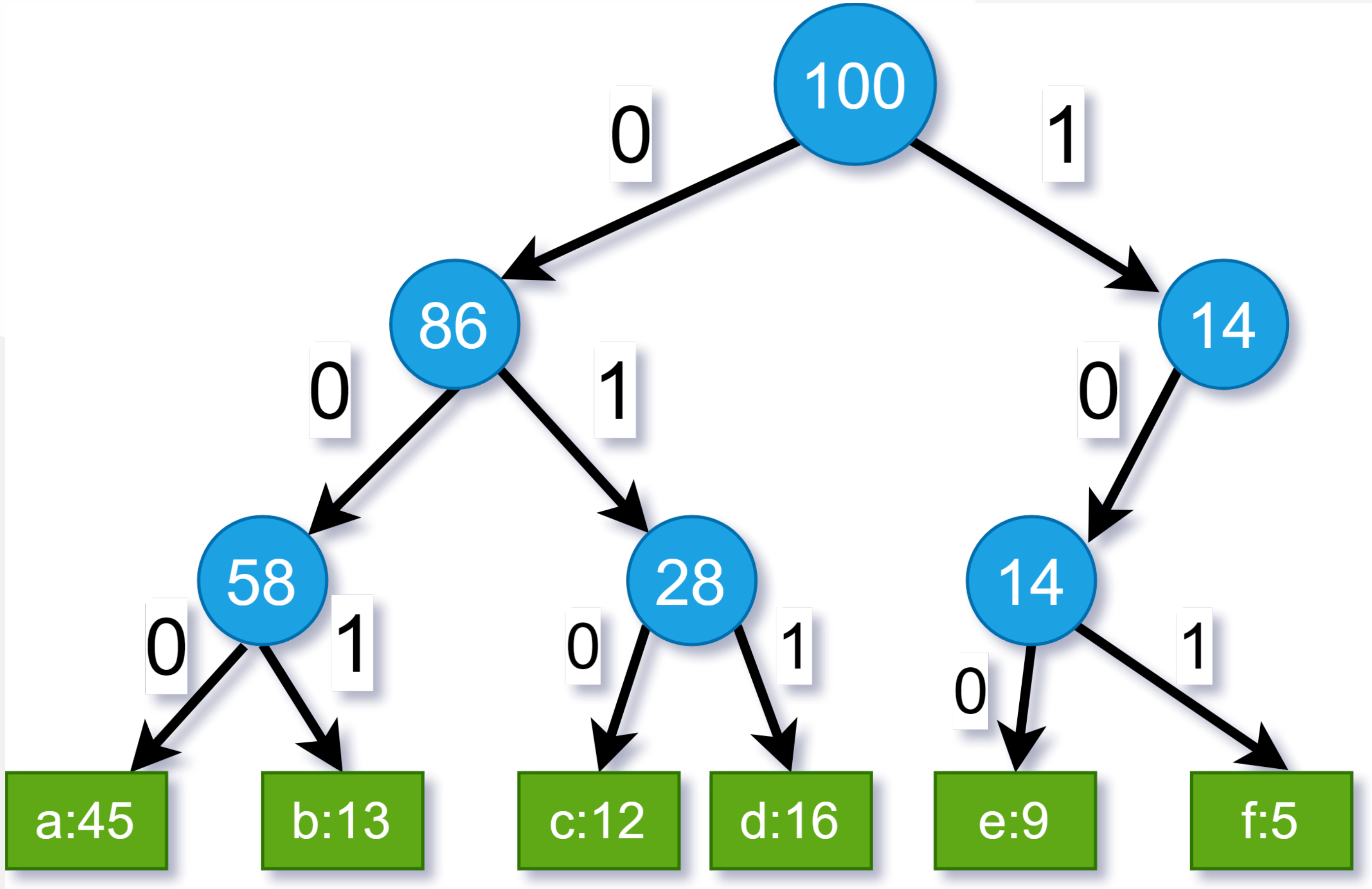
- Example: Decode encoded file 001011101
  - 001011101
  - 0.01011101
  - $0.0.1011101$
  - $0.0.101.1101$
  - $0.0.101.1101$
  - $aabe$

Prefix Codes

- Convenient representation for the prefix code:
  - a binary tree whose leaves are the given characters
- Binary codeword for a character is the path from the root to that character in the binary tree
- "0" means "**go to the left child**"
- "1" means "**go to the right child**"

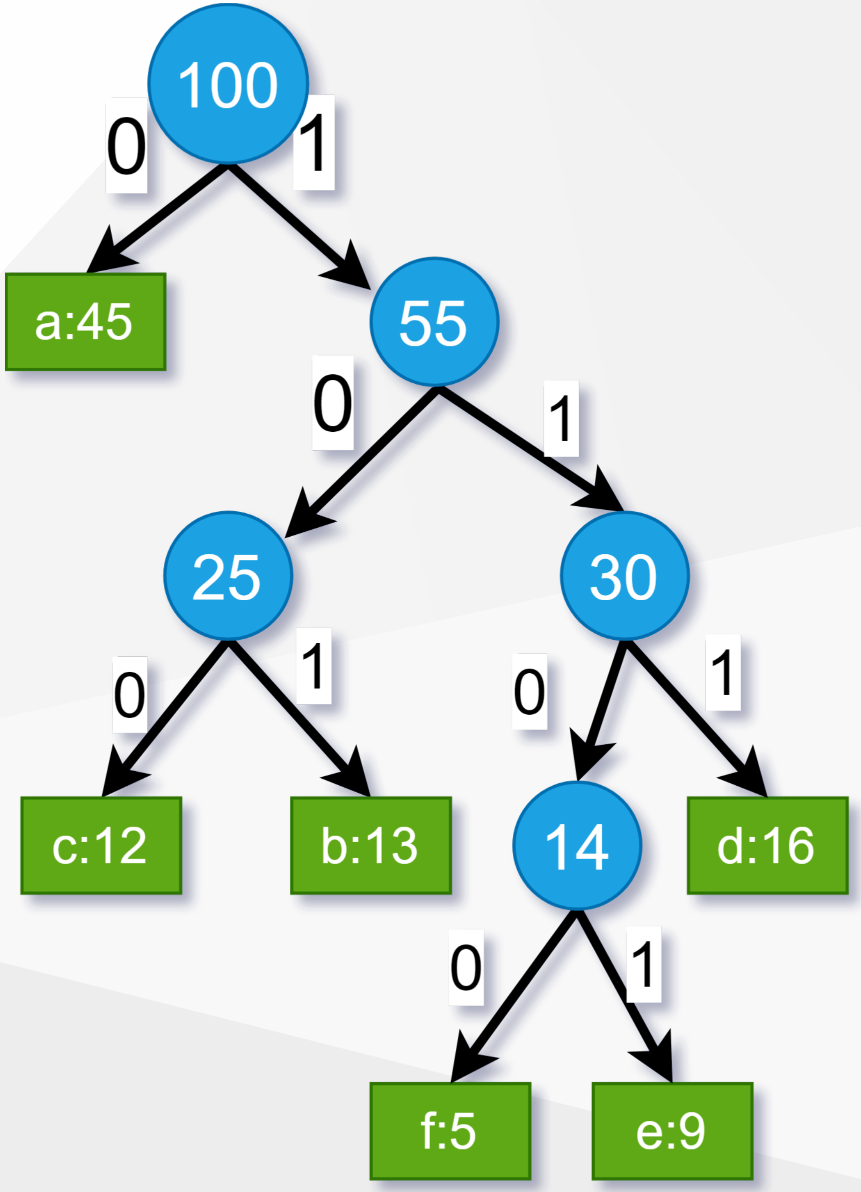
Binary Tree Representation of Prefix Codes

- **Weight of an internal node:** sum of weights of the leaves in its subtree
- The binary tree corresponding to the fixed-length code



Binary Tree Representation of Prefix Codes

- **Weight of an internal node:** sum of weights of the leaves in its subtree
- The binary tree corresponding to the **optimal variable-length** code
- An optimal code for a file is always represented by a **full binary tree**



Full Binary Tree Representation of Prefix Codes

- Consider an **FBT** corresponding to an optimal prefix code
- It has $|C|$ leaves (external nodes)
- One for each letter of the alphabet where C is the alphabet from which the characters are drawn
- **Lemma:** An **FBT** with $|C|$ external nodes has exactly $|C| - 1$ internal nodes

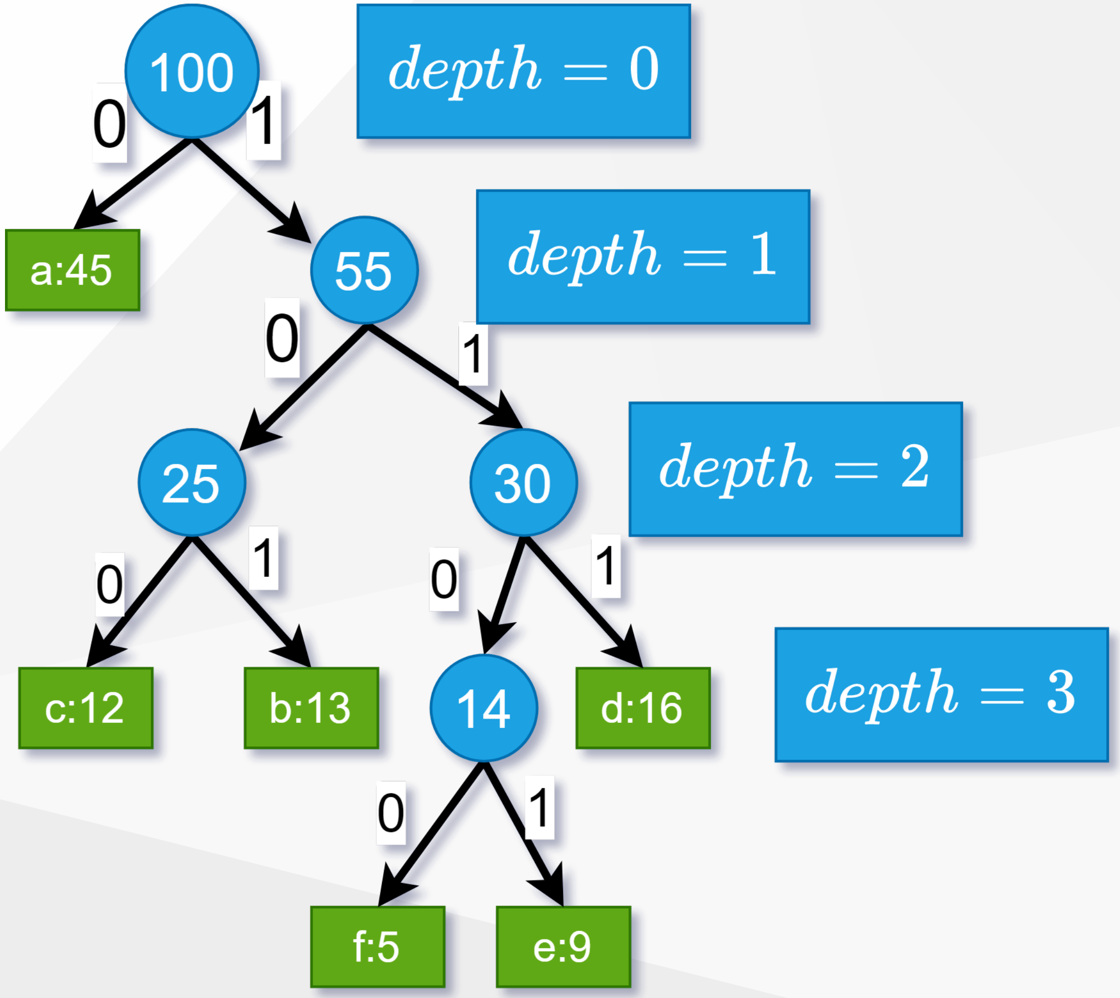
Full Binary Tree Representation of Prefix Codes

- Consider an $FBT\ T$, corresponding to a prefix code.
- **Notation**:
  - $f(c)$: frequency of character c in the file
  - $d_T(c)$: depth of c's leaf in the $FBT\ T$
  - $B(T)$: the number of bits required to encode the file
- What is the length of the codeword for c?
  - $d_T(c)$, same as the depth of c in T
- How to compute $B(T)$, cost of tree T?
  - $B(T) = \sum_{c \in C} f(c) d_T(c)$

Cost Computation - Example

$$B(T) = \sum_{c \in C} f(c) d_T(c)$$

$$\begin{aligned} B(T) = &(45 \times 1) + (12 \times 3) + \\ &(13 \times 3) + (16 \times 3) + \\ &(5 \times 4) + (9 \times 4) \\ =&224 \end{aligned}$$



Prefix Codes

- **Lemma:** Let each internal node i is labeled with the sum of the weight $w(i)$ of the leaves in its subtree
- Then

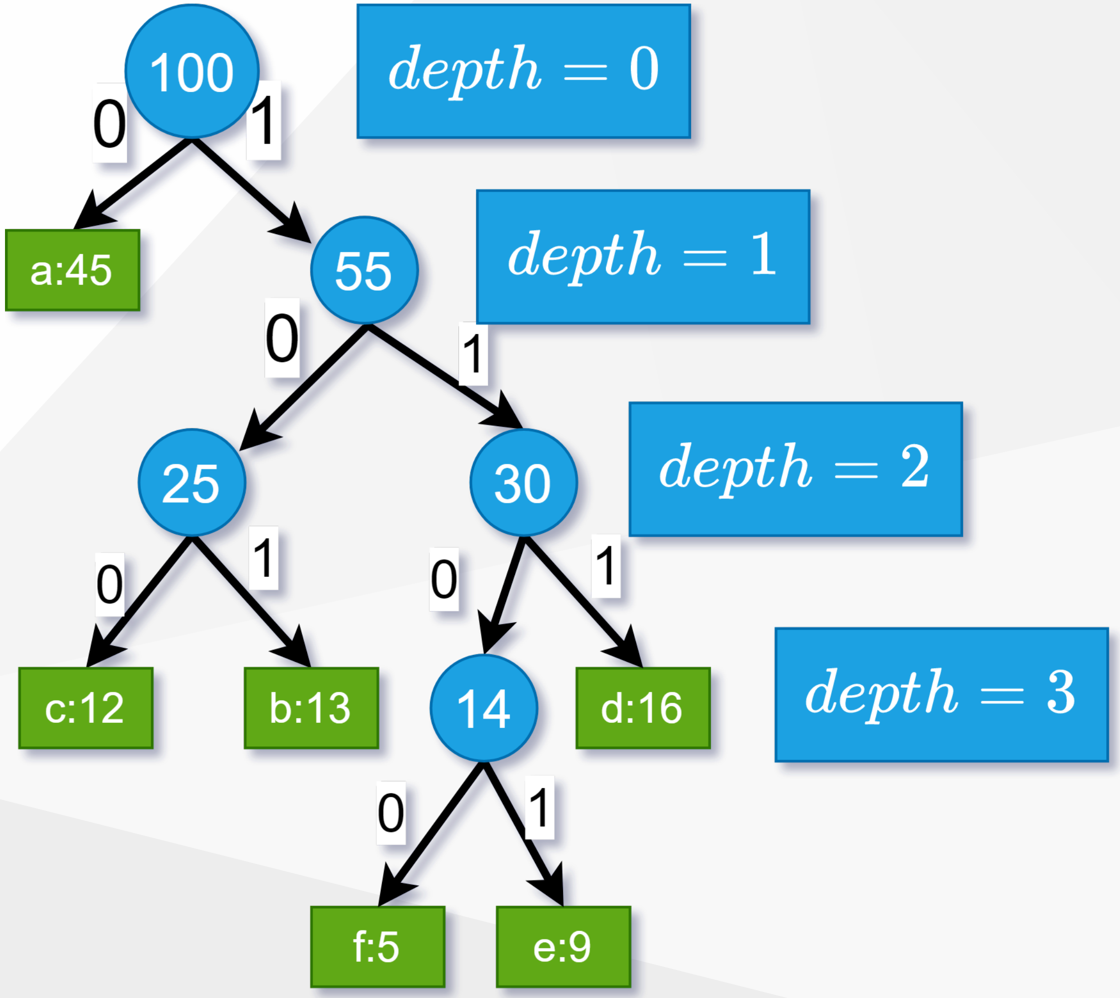
$$B(T) = \sum_{c \in C} f(c) d_T(c) = \sum_{i \in I_T} w(i)$$

- *where I_T is the set of internal nodes of T*
- **Proof:** Consider a leaf node c with $f(c)$ & $d_T(c)$
  - Then, $f(c)$ appears in the weights of $d_T(c)$ internal node
  - along the path from c to the root
  - Hence, $f(c)$ appears $d_T(c)$ times in the above summation

Cost Computation - Example

$$B(T) = \sum_{i \in I_T} w(i)$$

$$\begin{aligned} B(T) =& 100 + 55 + \\ & 25 + 30 + 14 \\ =& 224 \end{aligned}$$



Constructing a Huffman Code

- **Problem Formulation:** For a given character set C, construct an optimal prefix code with the minimum total cost
- **Huffman** invented a **greedy algorithm** that constructs an optimal prefix code called a **Huffman code**
- The greedy algorithm
  - builds the **FBT** corresponding to the optimal code in a **bottom-up** manner
  - begins with a set of $|C|$ leaves
  - performs a sequence of $|C|-1$ "**merges**" to create the final tree

Constructing a Huffman Code

- A **priority queue** Q, keyed on f, is used
  to identify the two **least-frequent** objects to merge
- The result of the **merger** of two objects is a **new object**
  - inserted into the priority queue according to its frequency
  - which is the sum of the frequencies of the two objects merged

Constructing a Huffman Code

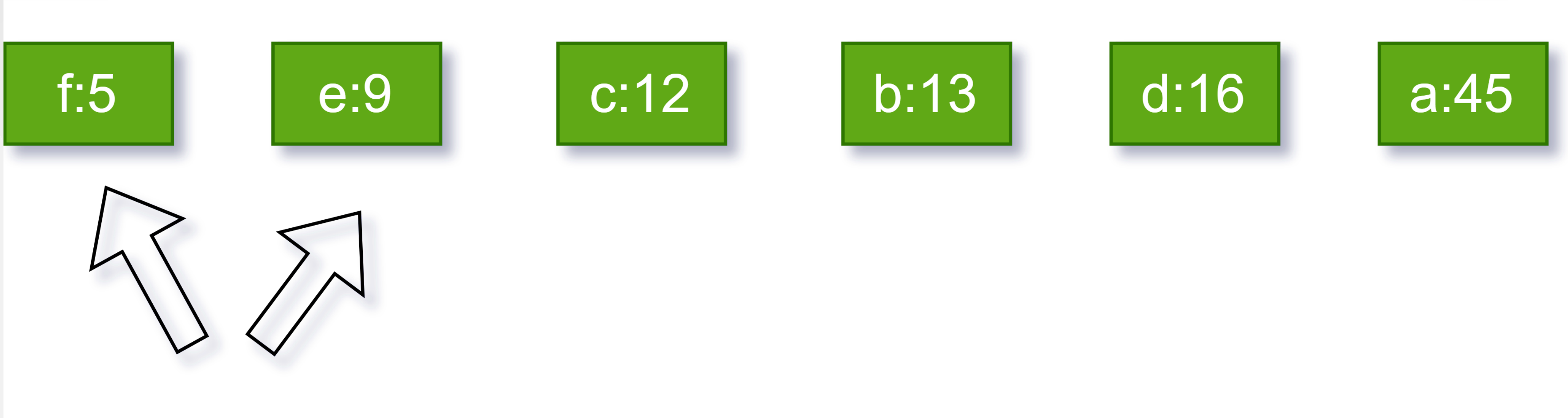
- Priority queue is implemented as a binary heap
- Initiation of Q (BUILD-HEAP): $O(n)$ time
- EXTRACT-MIN & INSERT take $O(lgn)$ time on Q with n objects

Constructing a Huffman Code

$$
\begin{array}{l}
\text{HUFFMAN}(c) \\
\quad n \leftarrow |C| \\
\quad Q \leftarrow \text{BUILD-HEAP}(c) \\
\quad for\ i \leftarrow 1\ to\ n-1\ do \\
\quad\quad z \leftarrow \text{ALLOCATE-NODE}() \\
\quad\quad x \leftarrow left[z] \leftarrow \text{EXTRACT-MIN}(Q) \\
\quad\quad y \leftarrow right[z] \leftarrow \text{EXTRACT-MIN}(Q) \\
\quad\quad f[z] \leftarrow f[x] \leftarrow f[y] \\
\quad\quad \text{INSERT}(Q, z) \\
\quad return\ \text{EXTRACT-MIN}(Q) \lhd \text{one object left in } Q
\end{array}
$$

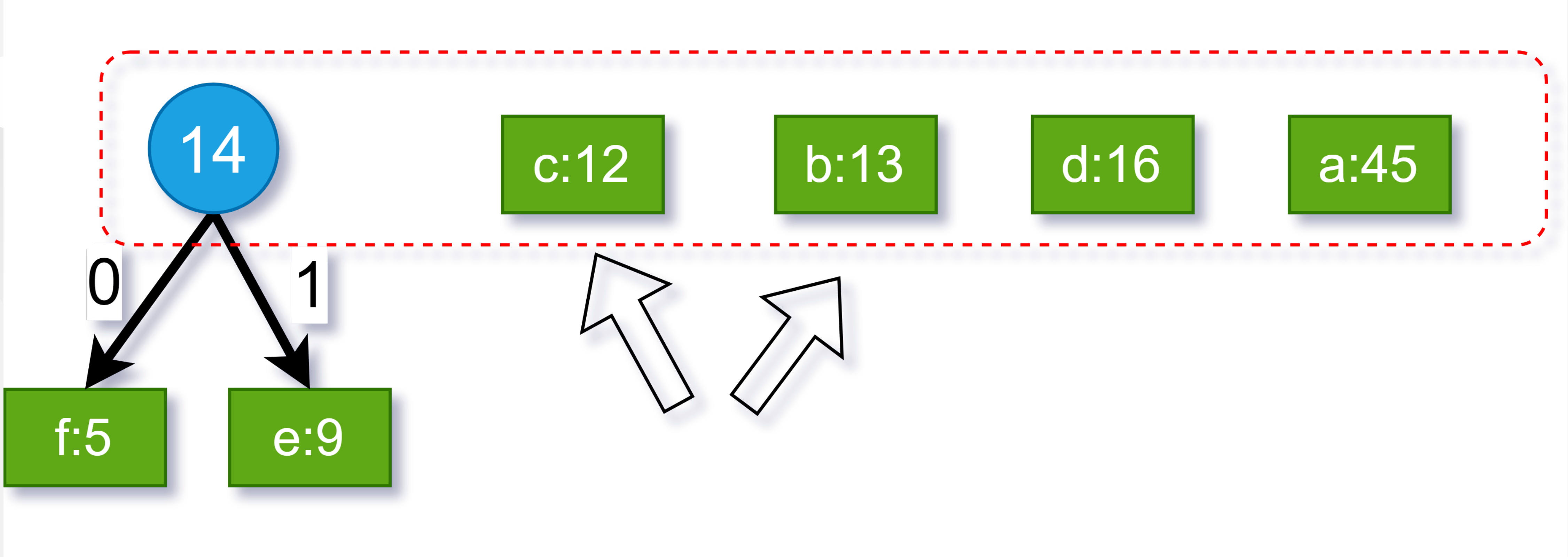
Constructing a Huffman Code - Example

- Start with one leaf node for each character
- The 2 nodes with the least frequencies: $f\&e$
- Merge $f\&e$ and create an internal node
- Set the internal node frequency to $5 + 9 = 14$

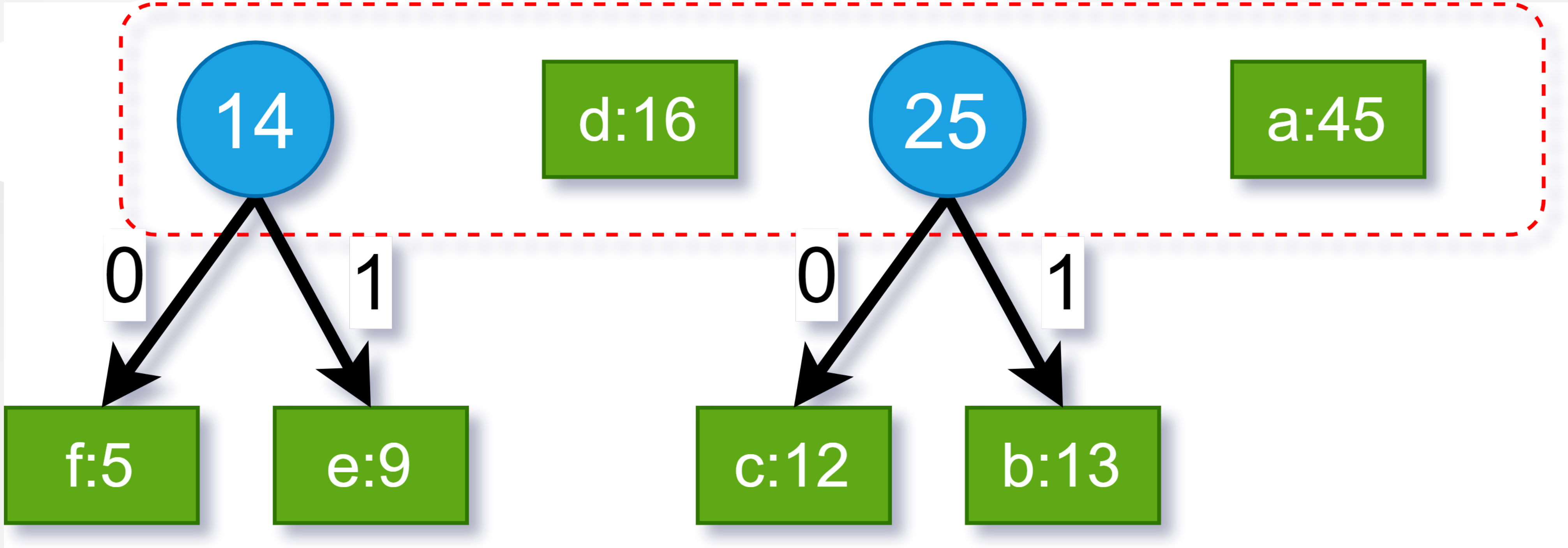


Constructing a Huffman Code - Example

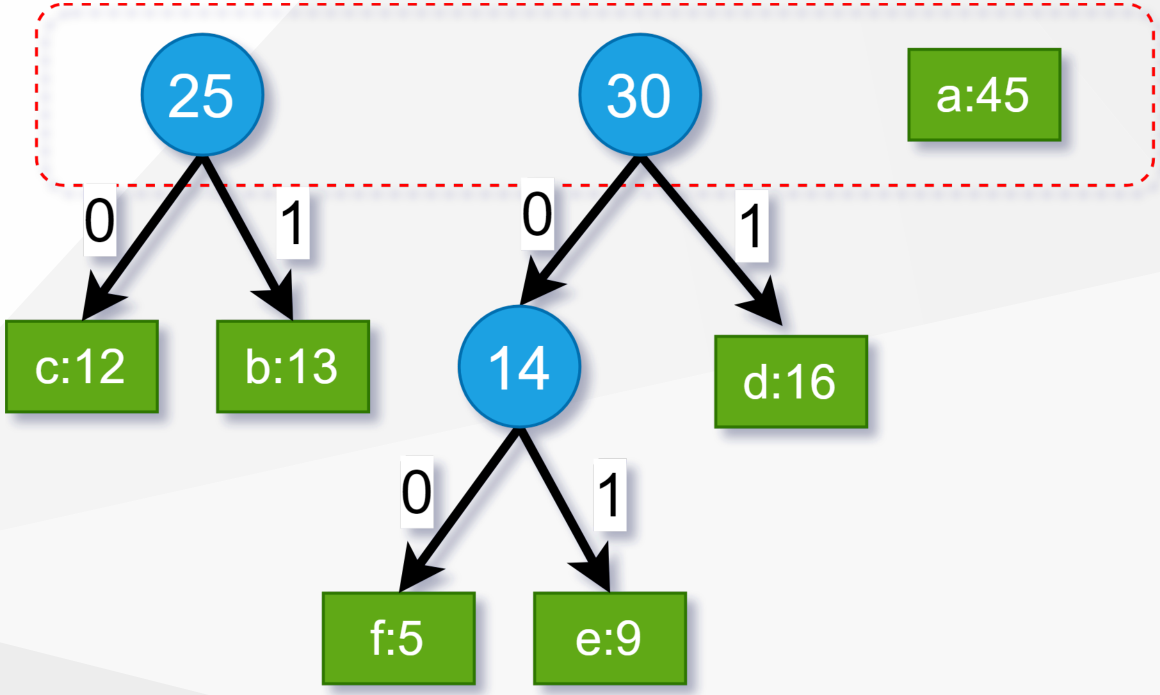
- The 2 nodes with least frequencies: $b \& c$



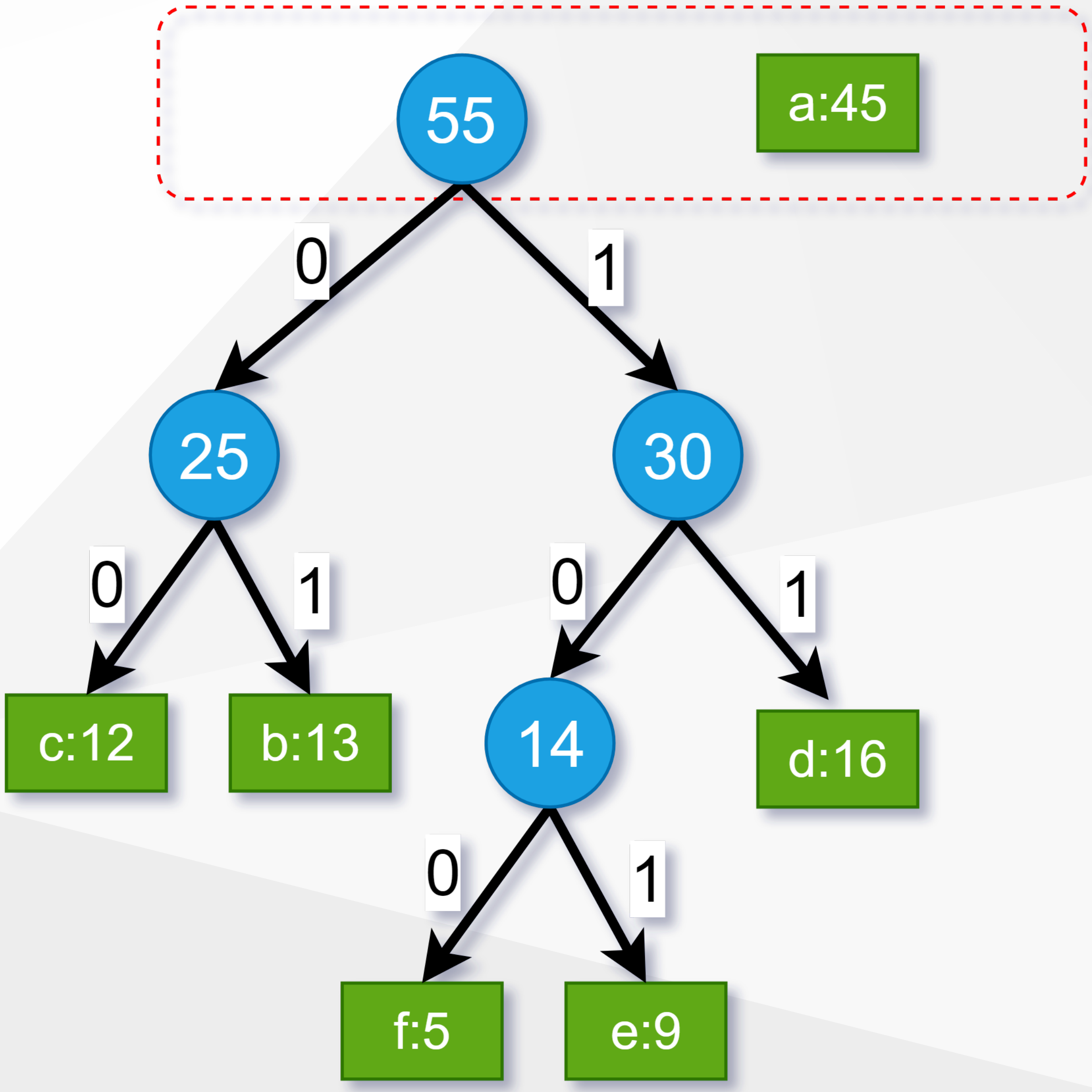
Constructing a Huffman Code - Example



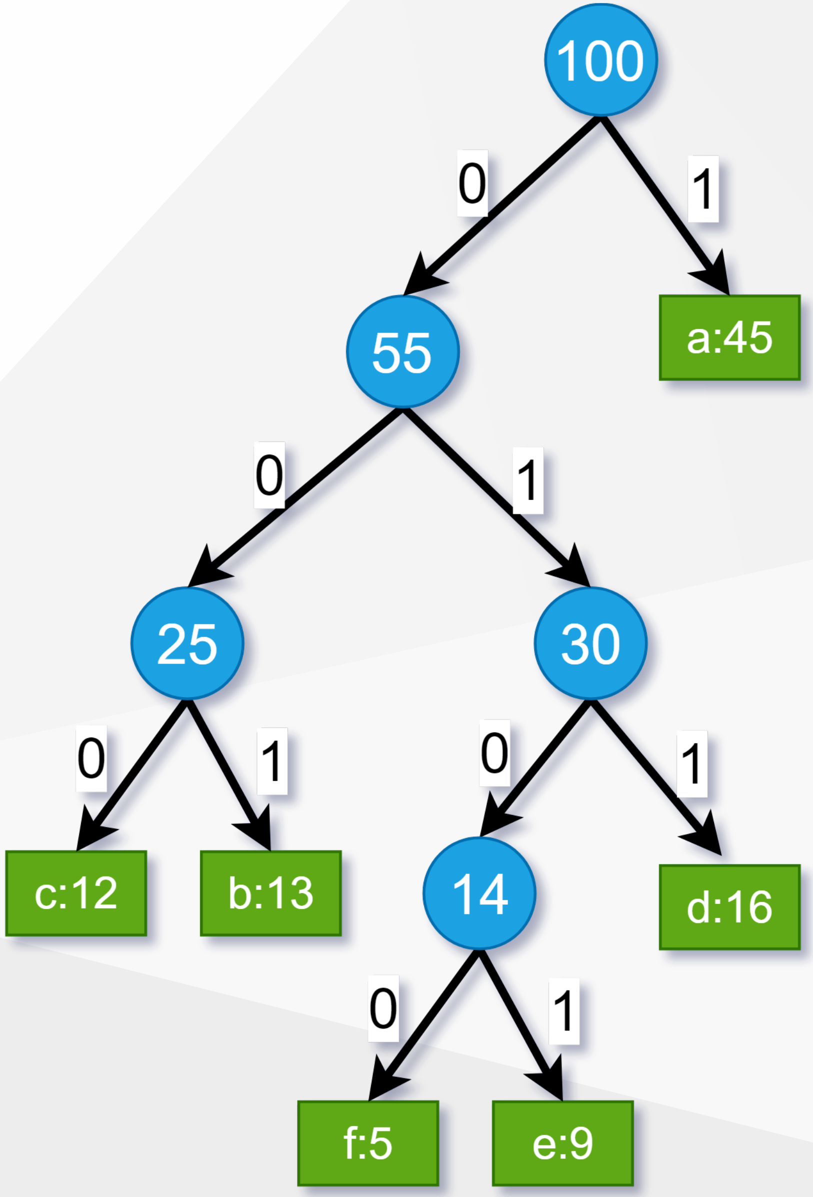
Constructing a Huffman Code - Example



Constructing a Huffman Code - Example



Constructing a Huffman Code - Example



Correctness Proof of Huffman's Algorithm

- **We need to prove:**
  - The greedy choice property
  - The optimal substructure property
- **What is the greedy step in Huffman's algorithm?**
  - *Merging the two characters with the lowest frequencies*
- *We will first prove the greedy choice property*

Greedy Choice Property

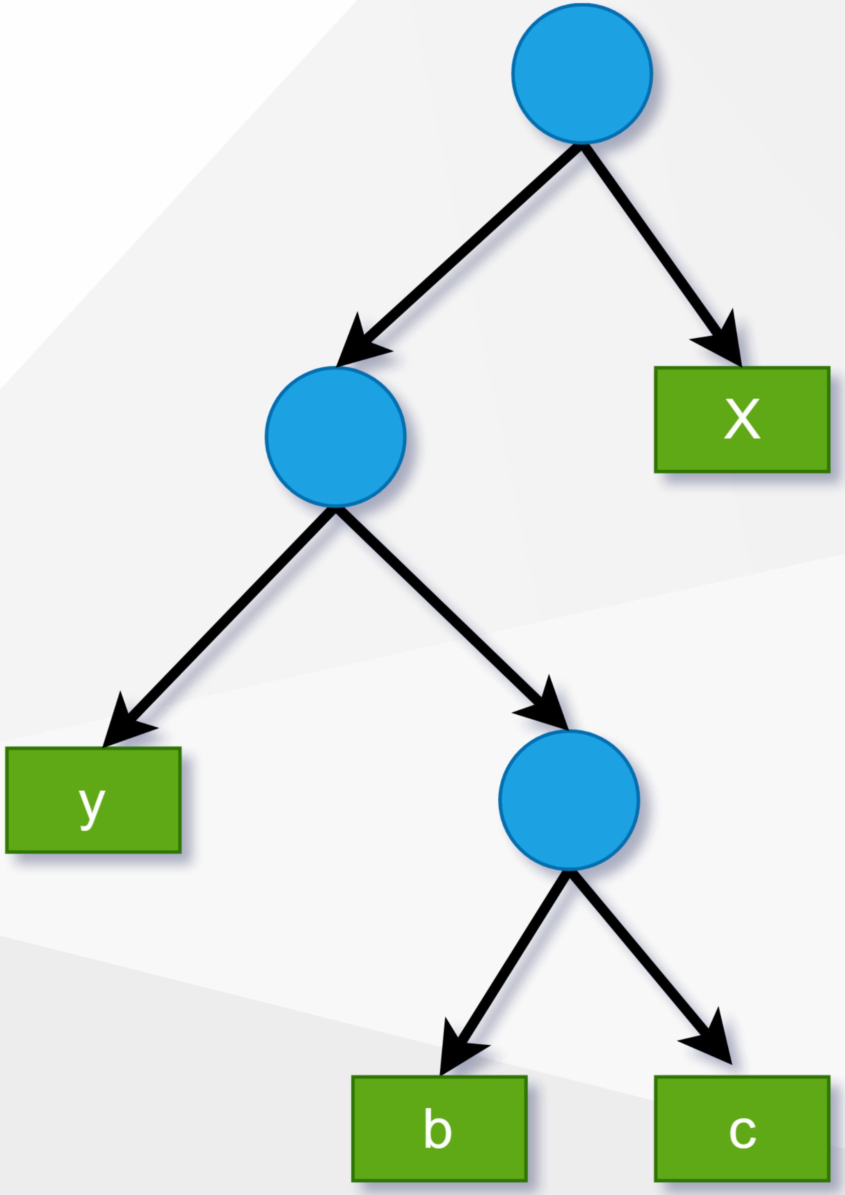
- **Lemma 1:** Let $x\&y$ be two characters in C having the **lowest frequencies**.
- Then, $\exists$ an optimal prefix code for C in which the codewords for $x\&y$ have the same length and differ only in the last bit
- **Note:** *If $x\&y$ are merged in Huffman's algorithm, their codewords are guaranteed to have the same length and they will differ only in the last bit*.
    - *Lemma 1* states that there exists an optimal solution where this is the case.

Greedy Choice Property - Proof

- Outline of the proof:
  - Start with an arbitrary optimal solution
  - Convert it to an optimal solution that satisfies the greedy choice property.
- **Proof:** Let T be an arbitrary optimal solution where:
  - $b\&c$ are the sibling leaves with the **max depth**
  - $x\&y$ are the characters with the **lowest frequencies**

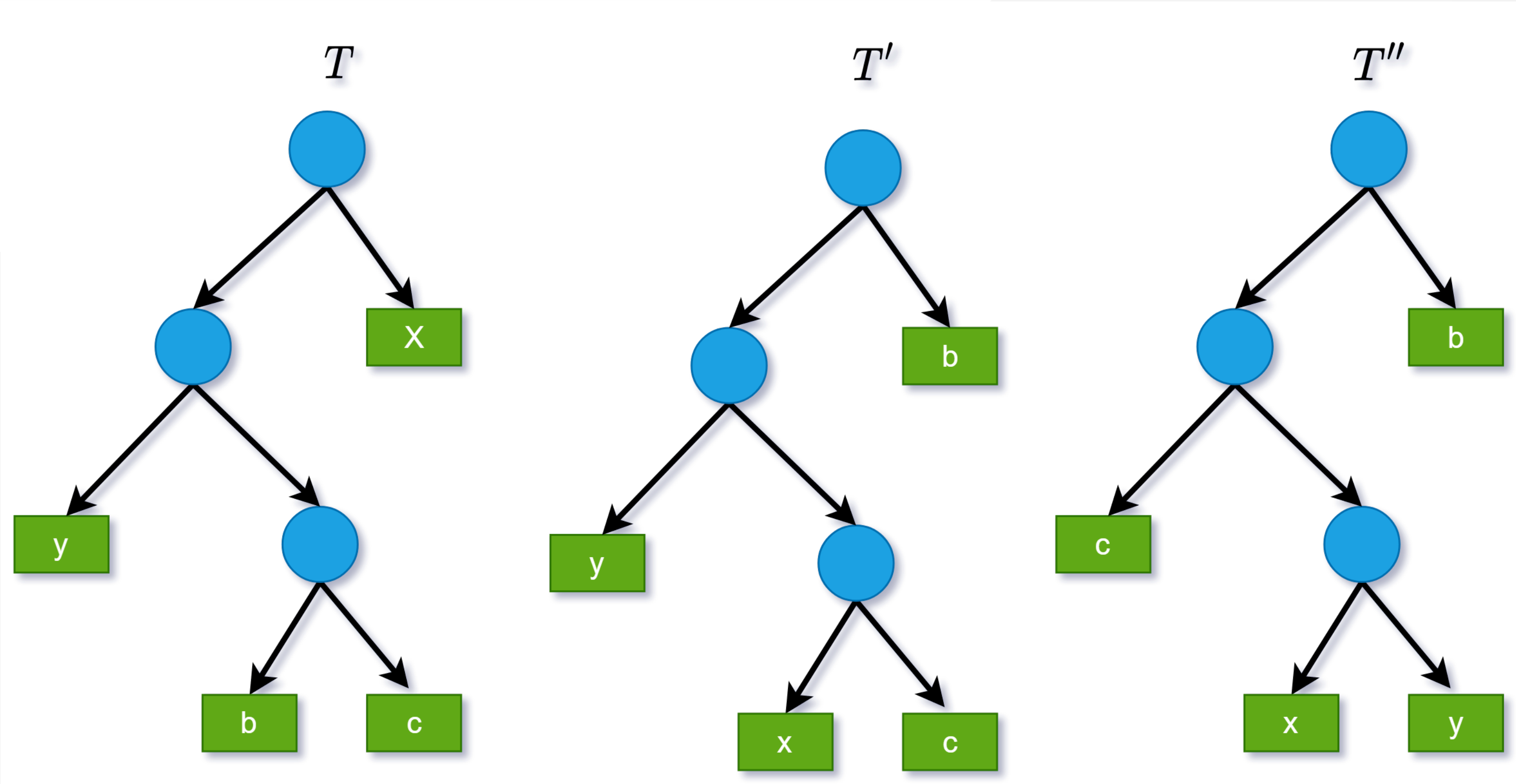
Greedy Choice Property - Proof

- Reminder:
  - $b\&c$ are the nodes with max depth
  - $x\&y$ are the nodes with min freq.
- Without loss of generality, assume:
  - $f(x) \leq f(y)$
  - $f(b) \leq f(c)$
- Then, it must be the case that:
  - $f(x) \leq f(b)$
  - $f(y) \leq f(c)$

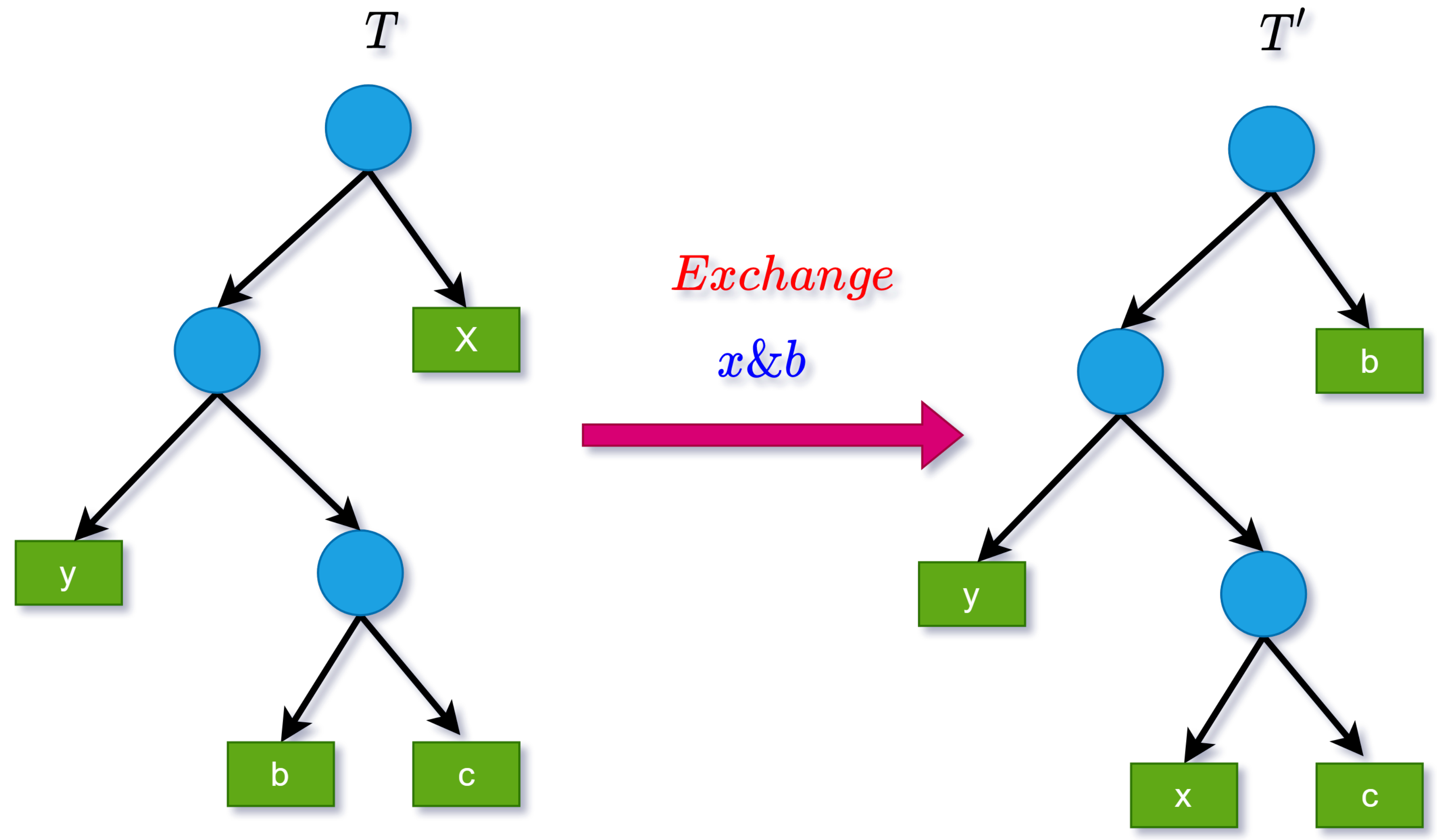


Greedy Choice Property - Proof

- $T \Rightarrow T'$: exchange the positions of the leaves $b\&x$
- $T' \Rightarrow T''$: exchange the positions of the leaves $c\&y$



Greedy Choice Property - Proof



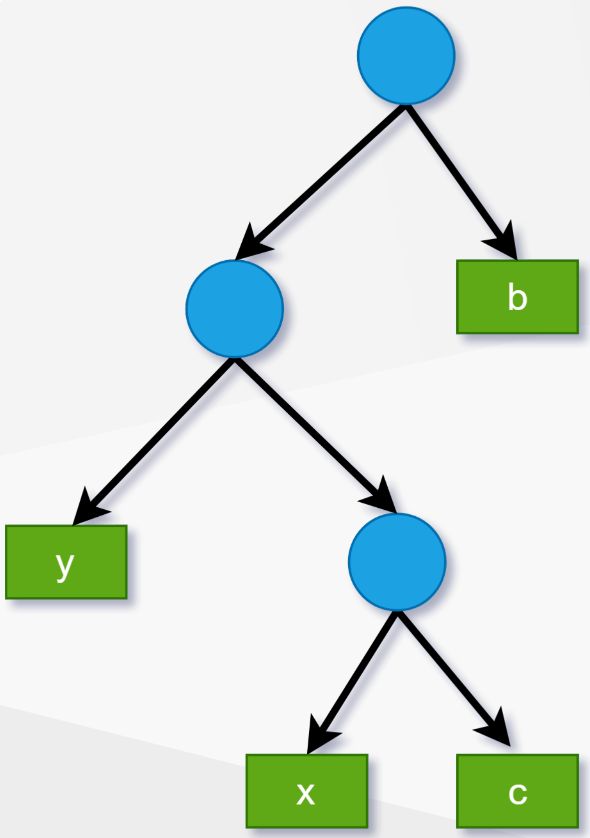
Greedy Choice Property - Proof

- **Reminder:** Cost of tree T'

$$B(T) = \sum_{c \in C} f(c) d_{T'}(c)$$

- How does $B(T')$ compare to $B(T)$?
- **Reminder:** $f(x) \leq f(b)$
  - $d_{T'}(x) = d_T(b)$ and $d_{T'}(b) = d_T(x)$

T'



Greedy Choice Property - Proof

- **Reminder:** $f(x) \leq f(b)$
  - $d_{T'}(x) = d_T(b)$ and $d_{T'}(b) = d_T(x)$
- The difference in cost between T and T':

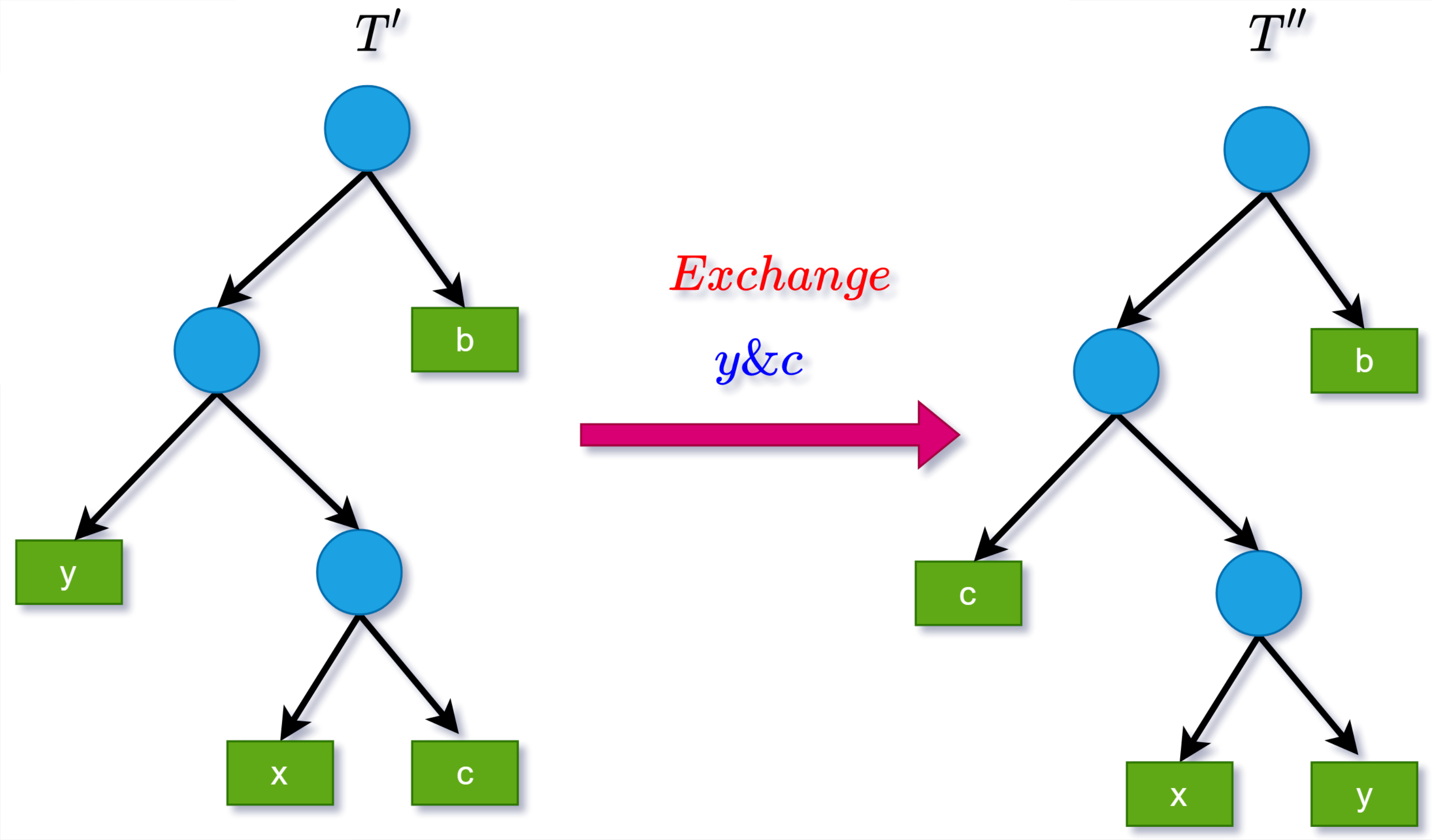
$$
\begin{aligned}
B(T) - B(T') &= \sum_{c \in C} f(c) d_T(c) - \sum_{c \in C} f(c) d_{T'}(c) \\
&= f[x] d_T(x) + f[b] d_T(b) - f[x] d_{T'}(x) - f[b] d_{T'}(b) \\
&= f[x] d_T(x) + f[b] d_T(b) - f[x] d_T(x) - f[b] d_T(b) \\
&= f[b](d_T(b) + d_T(x)) - f[x](d_T(b) - d_T(x)) \\
&= (f[b] - f[x])(d_T(b) + d_T(x))
\end{aligned}
$$

Greedy Choice Property - Proof

$$B(T) - B(T') = (f[b] - f[x])(d_T(b) + d_T(x))$$

- Since $f[b] - f[x] \geq 0$ and $d_T(b) \geq d_T(x)$
  - therefore $B(T') \leq B(T)$
- In other words, T' is also optimal

Greedy Choice Property - Proof



Greedy Choice Property - Proof

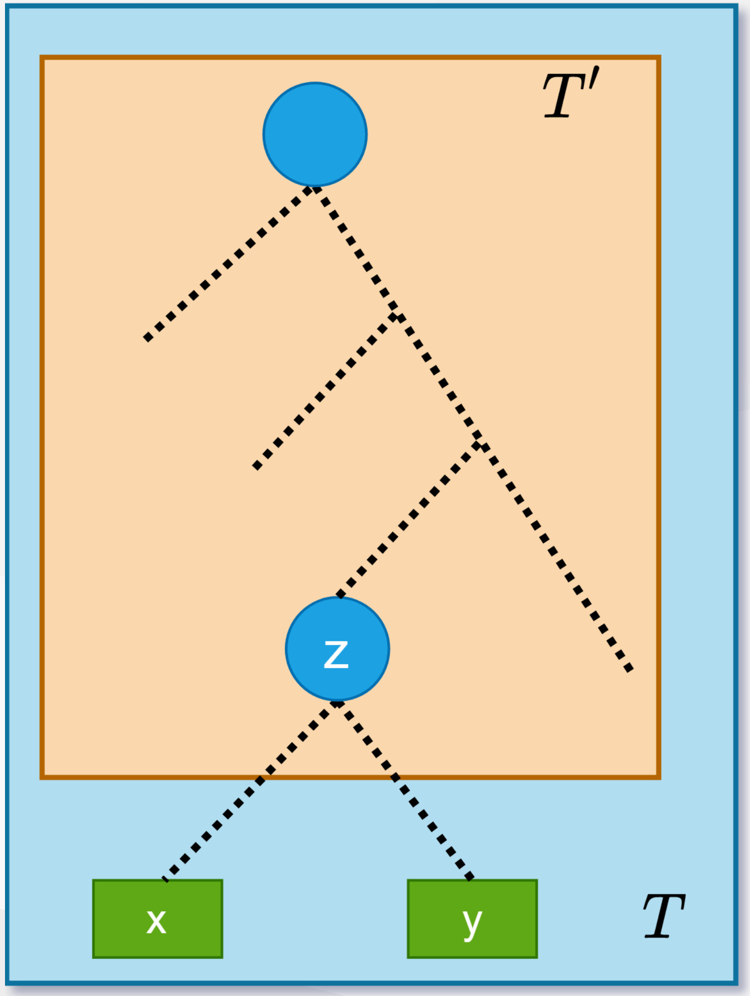
- We can similarly show that
- $B(T') - B(T'') \geq 0 \Rightarrow B(T'') \leq B(T')$
  - which implies $B(T'') \leq B(T)$
- Since T is optimal $\Rightarrow B(T'') = B(T) \Rightarrow T''$ is also optimal
- **Note:** T'' contains our greedy choice:
  - Characters $x \& y$ appear as sibling leaves of max-depth in T''
- Hence, the proof for the greedy choice property is complete

Greedy-Choice Property of Determining an Optimal Code

- **Lemma 1** implies that
  - process of building an optimal tree
  - by mergers can begin with the greedy choice of merging
  - those two characters with the lowest frequency
- We have already proved that $B(T) = \sum_{i \in I_T} w(i)$, that is,
  - the total cost of the tree constructed
  - is the **sum** of the **costs** of its **mergers** (**internal nodes**) **of all possible mergers**
- At each step **Huffman chooses** the merger that incurs the **least cost**

Optimal Substructure Property

- Consider an optimal solution T for alphabet C. Let x and y be any two sibling leaf nodes in T. Let z be the parent node of x and y in T.
- Consider the subtree T' where $T' = T - \{x, y\}$.
  - Here, consider z as a new character, where
    - $f[z] = f[x] + f[y]$
- **Optimal substructure property:** T' must be optimal for the alphabet C',
  where $C' = C - \{x, y\} \cup \{z\}$



Optimal Substructure Property - Proof

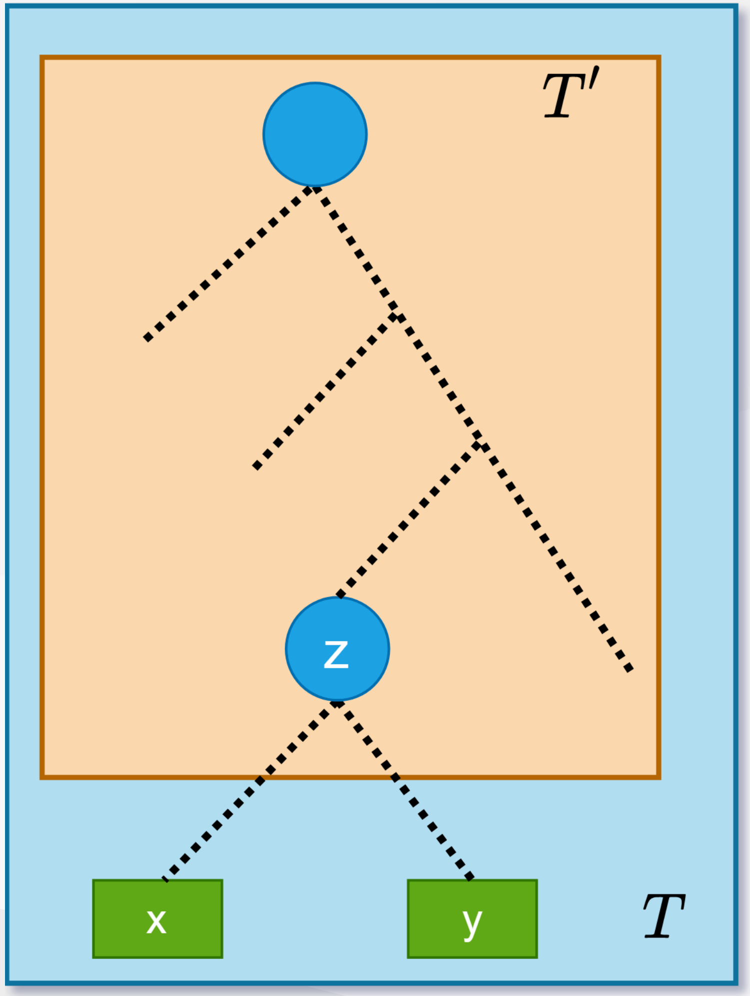
Reminder:

$$B(T) = \sum_{c \in C} f[c] d_T(c)$$

Try to express $B(T)$ in terms of $B(T')$.

Note: All characters in C' have the same depth in T and T'.

$$B(T) = B(T') - cost(z) + cost(x) + cost(y)$$



Optimal Substructure Property - Proof

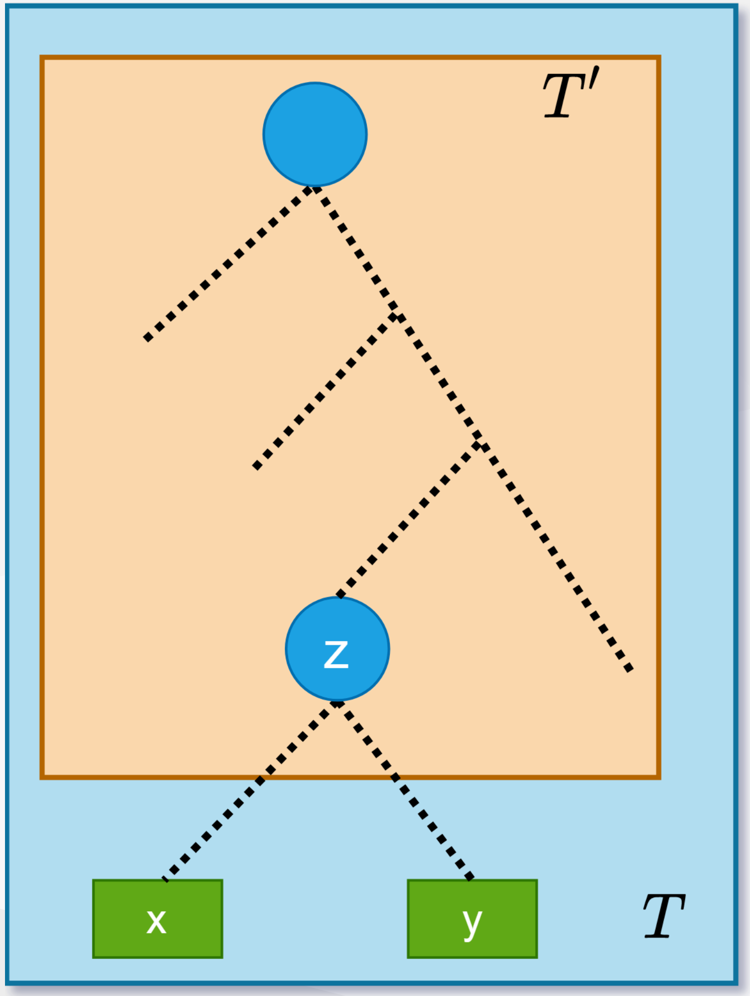
Reminder:

$$B(T) = \sum_{c \in C} f[c] d_T(c)$$

$$\begin{aligned} B(T) &= B(T') - cost(z) + cost(x) + cost(y) \\ &= B(T') - f[z].d_T(z) + f[x].d_T(x) + f[y].d_T(y) \\ &= B(T') - f[z].d_T(z) + (f[x] + f[y])(d_T(z) + 1) \\ &= B(T') - f[z].d_T(z) + f[z](d_T(z) + 1) \\ &= B(T') - f[z] \end{aligned}$$

$$d_T(x) = d_T(z) + 1$$
$$d_T(y) = d_T(z) + 1$$

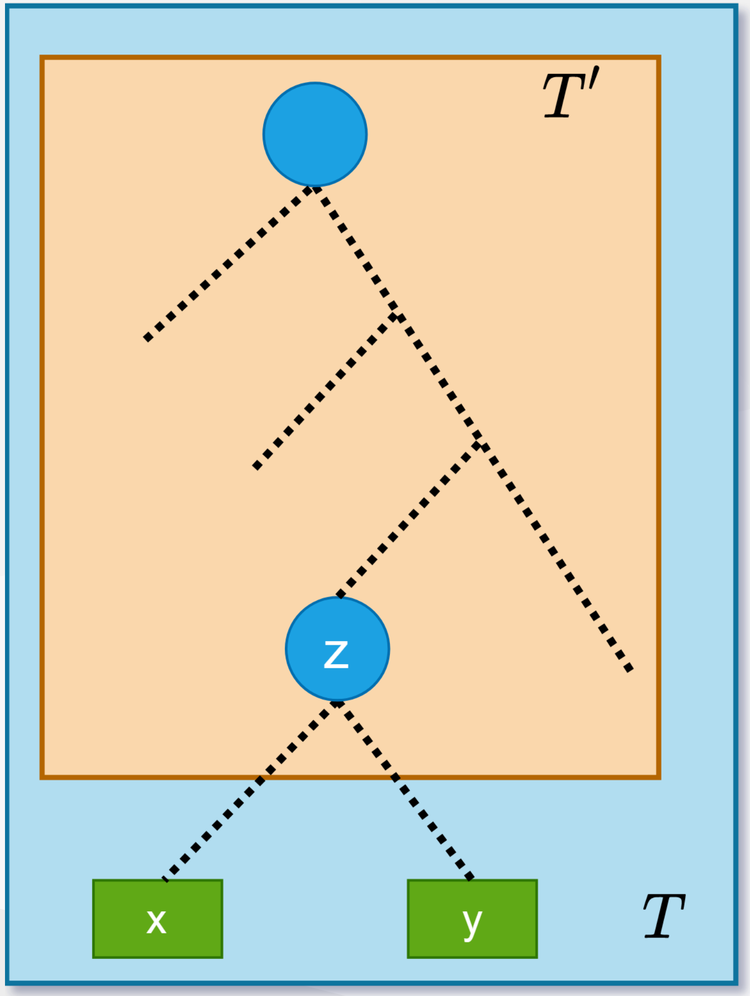
$$B(T) = B(T') + f[x] + f[y]$$



Optimal Substructure Property - Proof

- We want to prove that T' is optimal for
  - $C' = C - \{x, y\} \cup \{z\}$
- Assume by contradiction that that there exists another solution for C' with smaller cost than T'. Call this solution R':
- $B(R') < B(T')$
- Let us construct another prefix tree R by adding $x \& y$ as children of z in R'

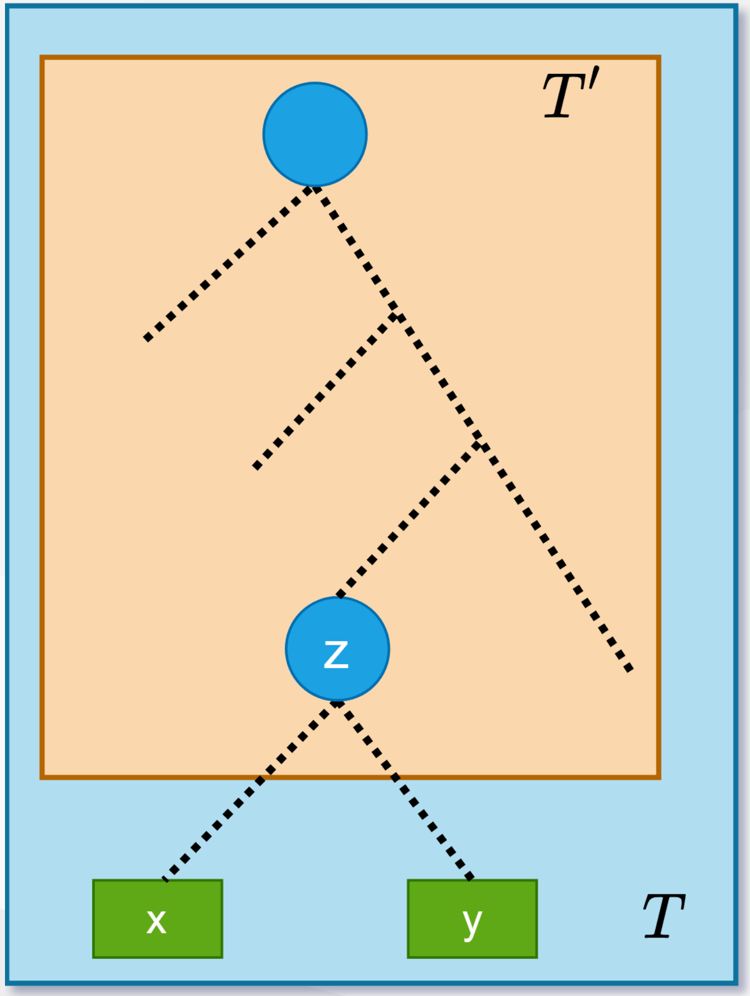
$$B(T) = B(T') + f[x] + f[y]$$



Optimal Substructure Property - Proof

- Let us construct another prefix tree R by adding $x\&y$ as children of z in R'.
- We have:
  - $B(R) = B(R') + f[x] + f[y]$
- In the beginning, we assumed that:
  - $B(R') < B(T')$
- So, we have:
  - $B(R) < B(T') + f[x] + f[y] = B(T)$

Contradiction! Proof complete



Greedy Algorithm for Huffman Coding - Summary

- For the greedy algorithm, we have proven that:
  - **The greedy choice property** holds.
  - **The optimal substructure property** holds.
- So, the greedy algorithm is optimal.

References

- Introduction to Algorithms, Third Edition | The MIT Press
- Bilkent CS473 Course Notes (new)
- Bilkent CS473 Course Notes (old)

$$-End - Of - Week - 9 - Course - Module-$$