CE100 Algorithms and Programming II

Week-7 (Greedy Algorithms, Knapsack)

Spring Semester, 2021-2022

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Greedy Algorithms, Knapsack

Outline

- Greedy Algorithms and Dynamic Programming Differences
- Greedy Algorithms
    - Activity Selection Problem
    - Knapsack Problems
        - The 0-1 knapsack problem
        - The fractional knapsack problem

Activity Selection Problem

Activity Selection Problem

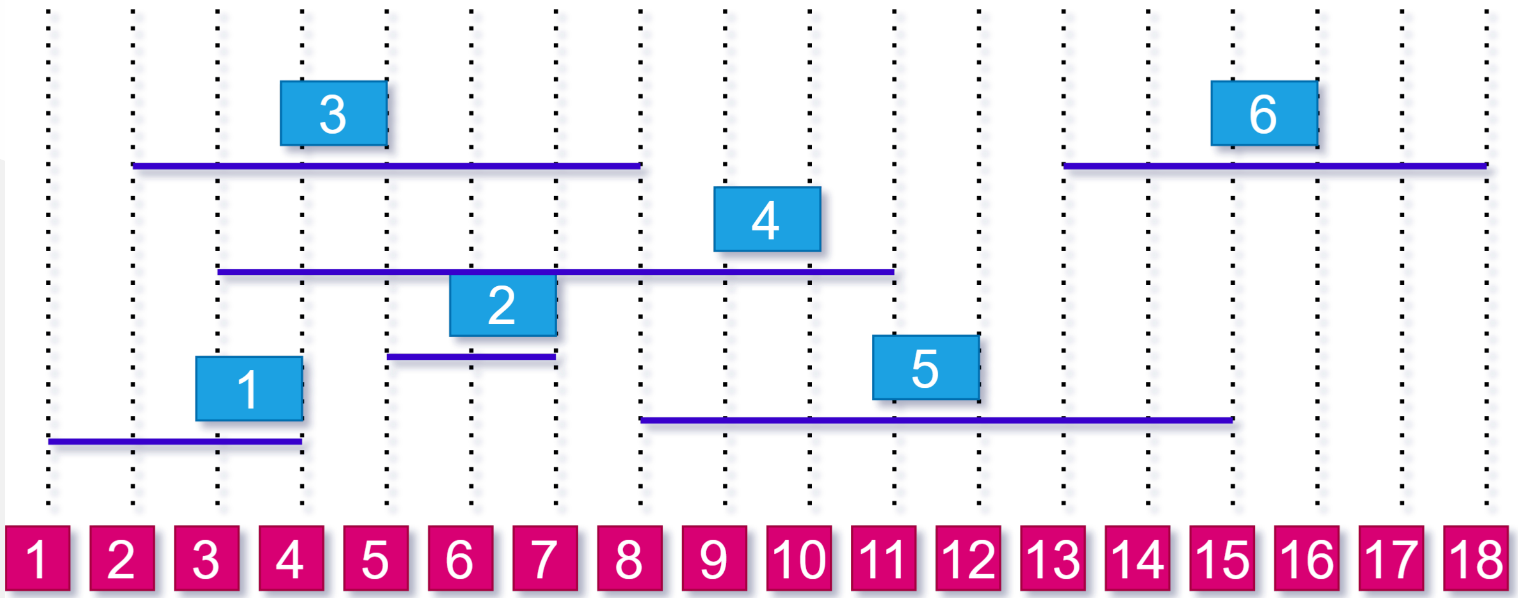
- We have:
    - A set of activities with fixed start and finish times
    - One shared resource (only one activity can use at any time)
- **Objective:** Choose the max number of compatible activities
- **Note:** Objective is to maximize the number of activities, not the total time of activities.
- **Example:**
    - *Activities:* Meetings with fixed start and finish times
    - *Shared resource:* A meeting room
        - *Objective:* Schedule the max number of meetings

Activity Selection Problem

- **Input:** a set $S = \{a_1, a_2, \dots, a_n\}$ of n activities
- s_i : Start time of activity a_i,
- f_i : Finish time of activity a_i
  Activity i takes place in $[s_i, f_i)$
- **Aim:** Find max-size subset A of mutually *compatible* activities
  - Max number of activities, not max time spent in activities
  - Activities i and j are compatible if intervals $[s_i, f_i)$ and $[s_j, f_j)$ do not overlap, i.e., either $s_i \geq f_j$ or $s_j \geq f_i$

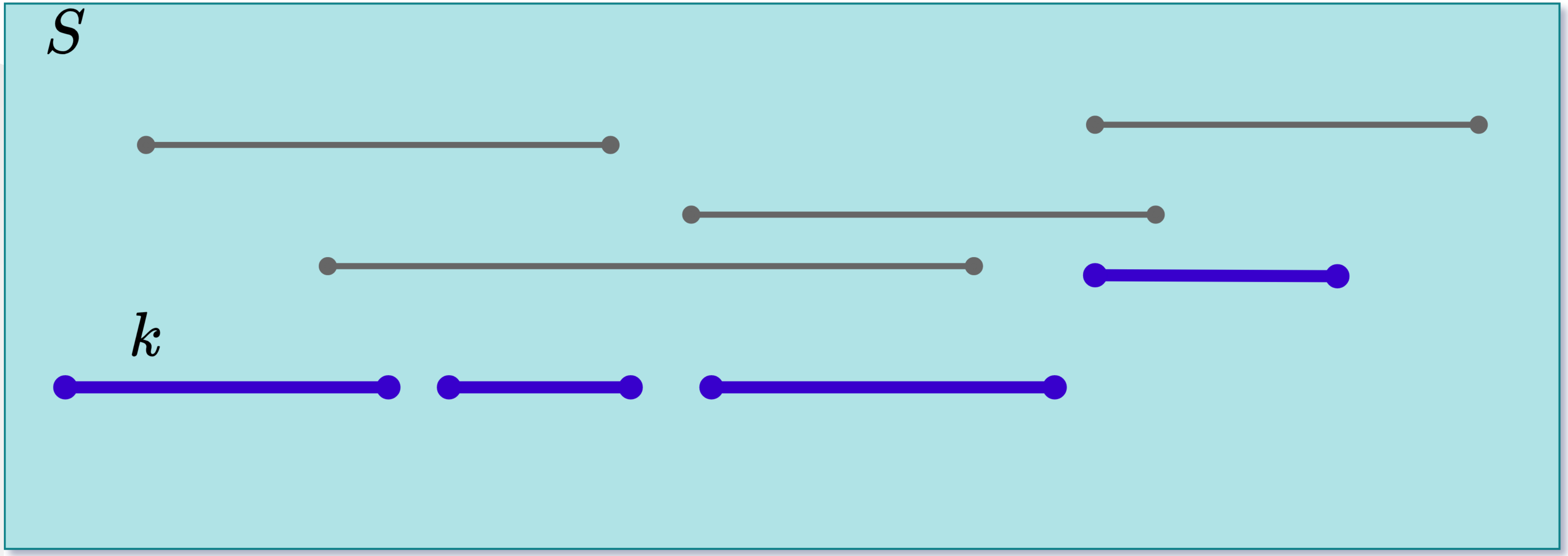
Activity Selection Problem An Example

$$S = [1,4), [5,7), [2,8), [3,11), [8,15), [13,18)$$



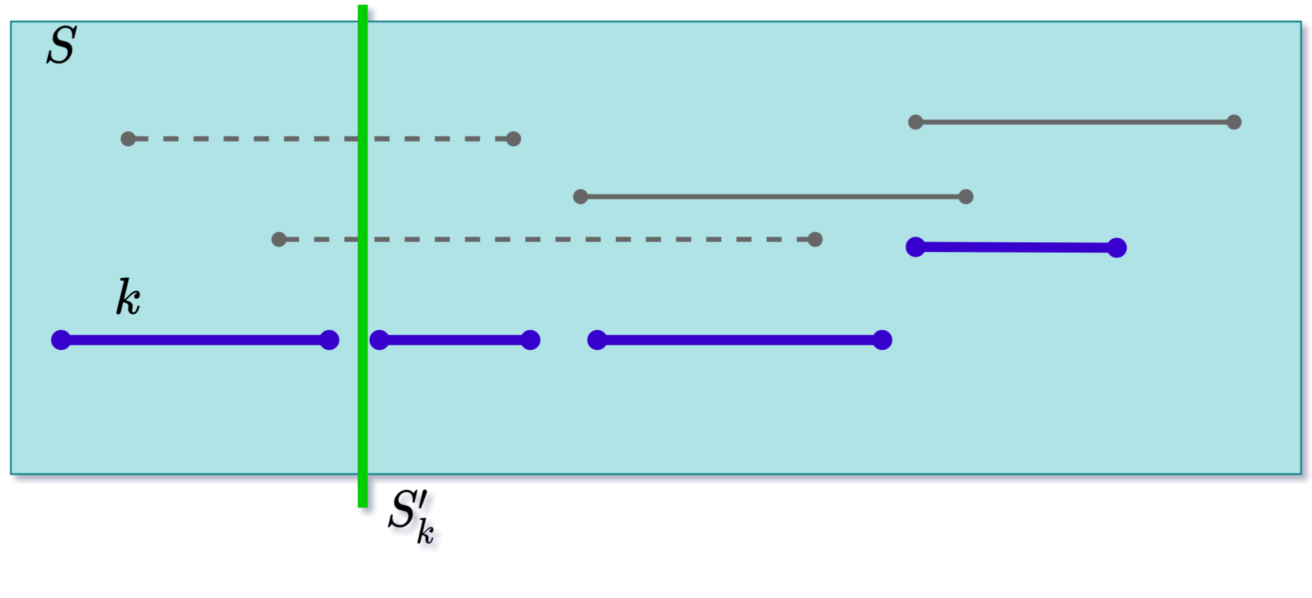
Optimal Substructure Property

- Consider an optimal solution A for activity set S.
- Let k be the activity in A with the **earliest finish time**



Optimal Substructure Property

- Consider an optimal solution A for activity set S.
- Let k be the activity in A with the **earliest finish time**
- Now, consider the **subproblem** S'_k that has the activities that start after k finishes, i.e. $S'_k = \{a_i \in S : s_i \geq f_k\}$
- What can we say about the optimal solution to S'_k ?



Optimal Substructure Property

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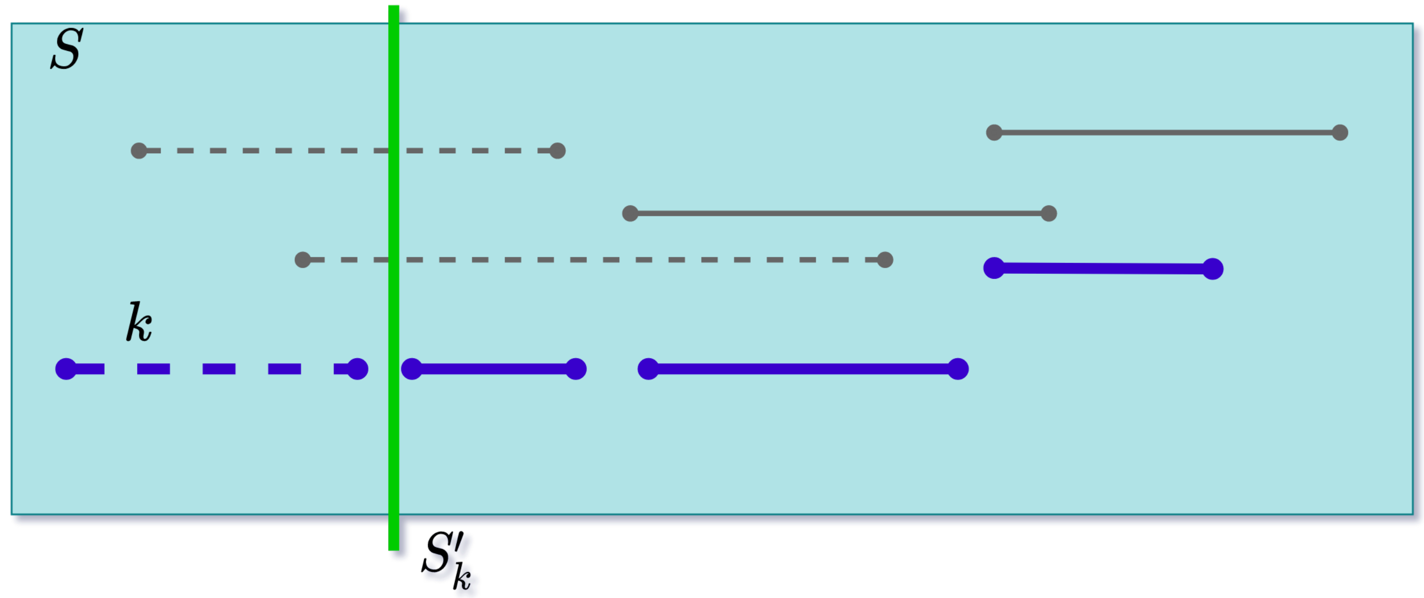
- $A - \{k\}$ is an optimal solution for S'_k. Why?



Optimal Substructure

- **Theorem:** Let k be the activity with the earliest finish time in an optimal soln $A \subseteq S$ then
  - $A - \{k\}$ is an optimal solution to subproblem
  - $S'_k = \{a_i \in S : s_i \geq f_k\}$
- **Proof (by contradiction):**
  - $\triangleright$ Let B' be an optimal solution to S'_k and
    - $|B'| > |A - \{k\}| = |A| - 1$
  - Then, $B = B' \cup \{k\}$ is compatible and
    - $|B| = |B'| + 1 > |A|$
  - Contradiction to the optimality of A

    $Q.E.D.$

Optimal Substructure

- **Recursive formulation:** Choose the first activity k, and then solve the remaining subproblem S'_k
- How to choose the first activity k?
    - DP, memoized recursion?
        - i.e. choose the k value that will have the max size for S'_k
- DP would work,
    - but is it necessary to try all possible values for k?

Greedy Choice Property

- Assume (without loss of generality) $f_1 \leq f_2 \leq \cdots \leq f_n$
  - If not, sort activities according to their finish times in non-decreasing order
- **Greedy choice property:** a sequence of locally optimal (greedy) choices $\Rightarrow$ an optimal solution
- How to choose the first activity **greedily** without losing optimality?

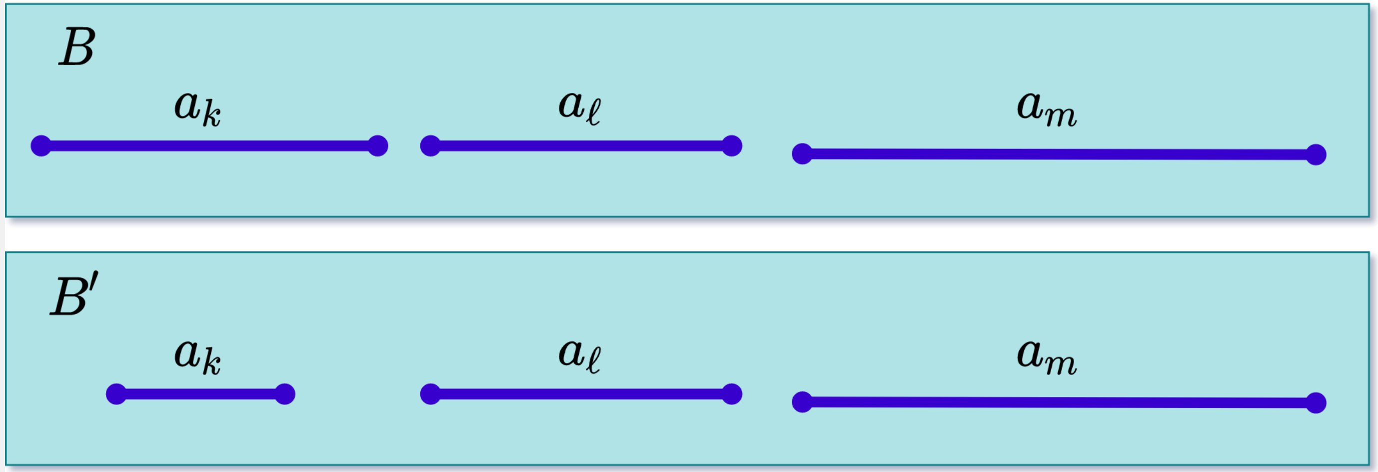
Greedy Choice Property - Theorem

- Let activity set $S = \{a_1, a_2, \ldots a_n\}$, where $f_1 \leq f_2 \leq \cdots \leq f_n$
- **Theorem:** There exists an optimal solution $A \subseteq S$ such that $a_1 \in A$

In other words, the activity with the earliest finish time is guaranteed to be in an optimal solution.

Greedy Choice Property - Proof

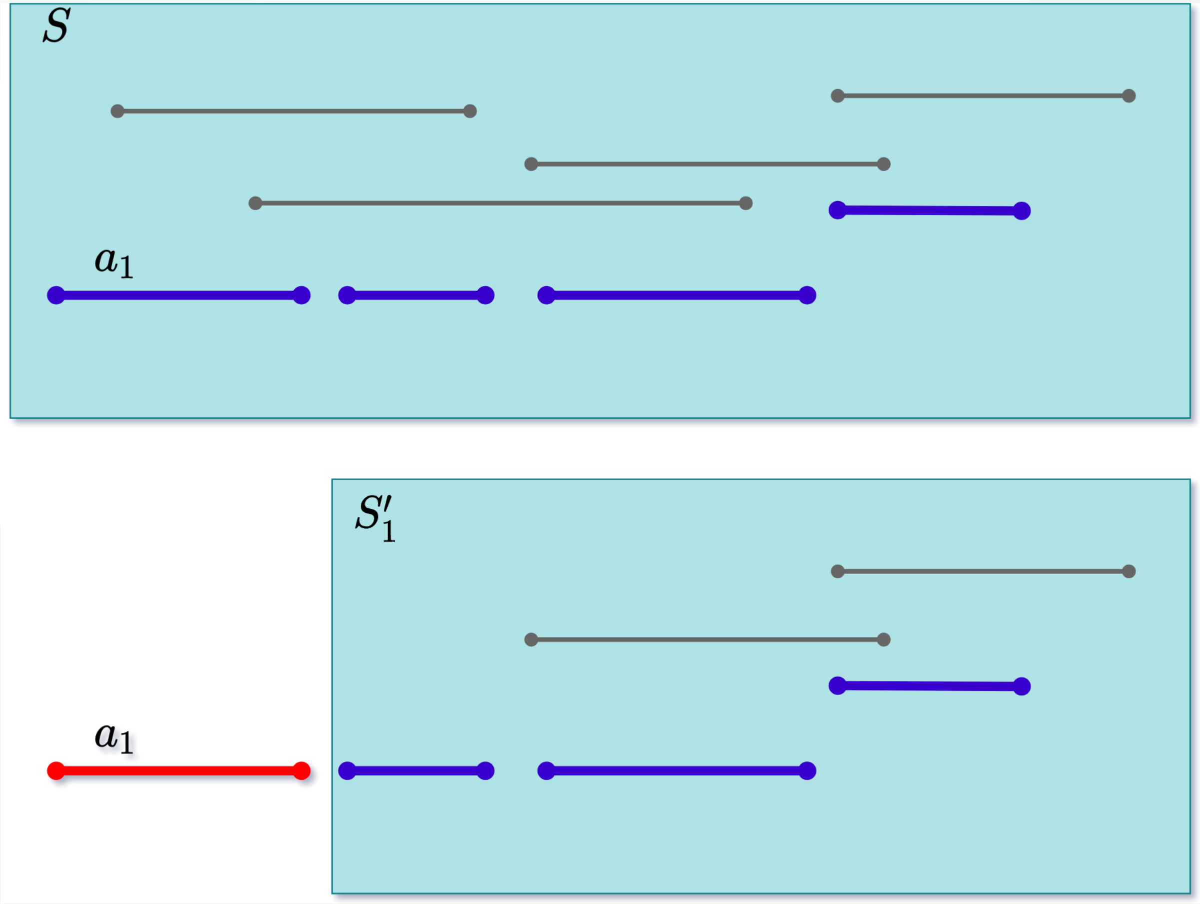
- **Theorem:** There exists an optimal solution $A \subseteq S$ such that $a_1 \in A$
- **Proof:** Consider an arbitrary optimal solution $B = \{a_k, a_\ell, a_m, \ldots\}$, where $f_k < f_\ell < f_m < \ldots$
  - If $k = 1$, then B starts with a_1, and the proof is complete
  - If $k > 1$, then create another solution B' by replacing a_k with a_1. Since $f_1 \leq f_k$, B' is guaranteed to be valid, and $|B'| = |B|$, hence also optimal



Greedy Algorithm

- So far, we have:
  - **Optimal substructure property:** If $A = \{a_k, \dots\}$ is an optimal solution, then $A - \{a_k\}$ must be optimal for subproblem $S'_{k'}$, where $Sk' = \{a_i \in S : s_i \geq f_k\}$
    - **Note:** a_k is the activity with the earliest finish time in A
  - **Greedy choice property:** There is an optimal solution A that contains a_1
    - **Note:** a_1 is the activity with the earliest finish time in S

Greedy Algorithm



explained in the next slide..

Greedy Algorithm

- **Theorem:** There exists an optimal solution $A \subseteq S$ such that $a_1 \in A$
- Basic idea of the greedy algorithm:
  - Add a_1 to A
  - Solve the remaining subproblem S_1', and then append the result to A
- **Remember arbitary optimal solution explaination from previous sections (finish time order is important for a_1 selection with star time and overlapping checking)**
  - $B = \{a_k, a_\ell, a_m, \dots\}$,
  - where $f_k < f_\ell < f_m < \dots$

Greedy Algorithm for Activity Selection

Definitions in Greedy Algorithm:

- j: specifies the index of most recent activity added to A
- $f_j = Max\{f_k : k \in A\}$, max finish time of any activity in A;
    - because activities are processed in non-decreasing order of finish times
- Thus, $s_i \geq f_j$ checks the compatibility of i to current A
- **Running time:** $\Theta(n)$ assuming that the activities were already sorted.

Greedy Algorithm for Activity Selection

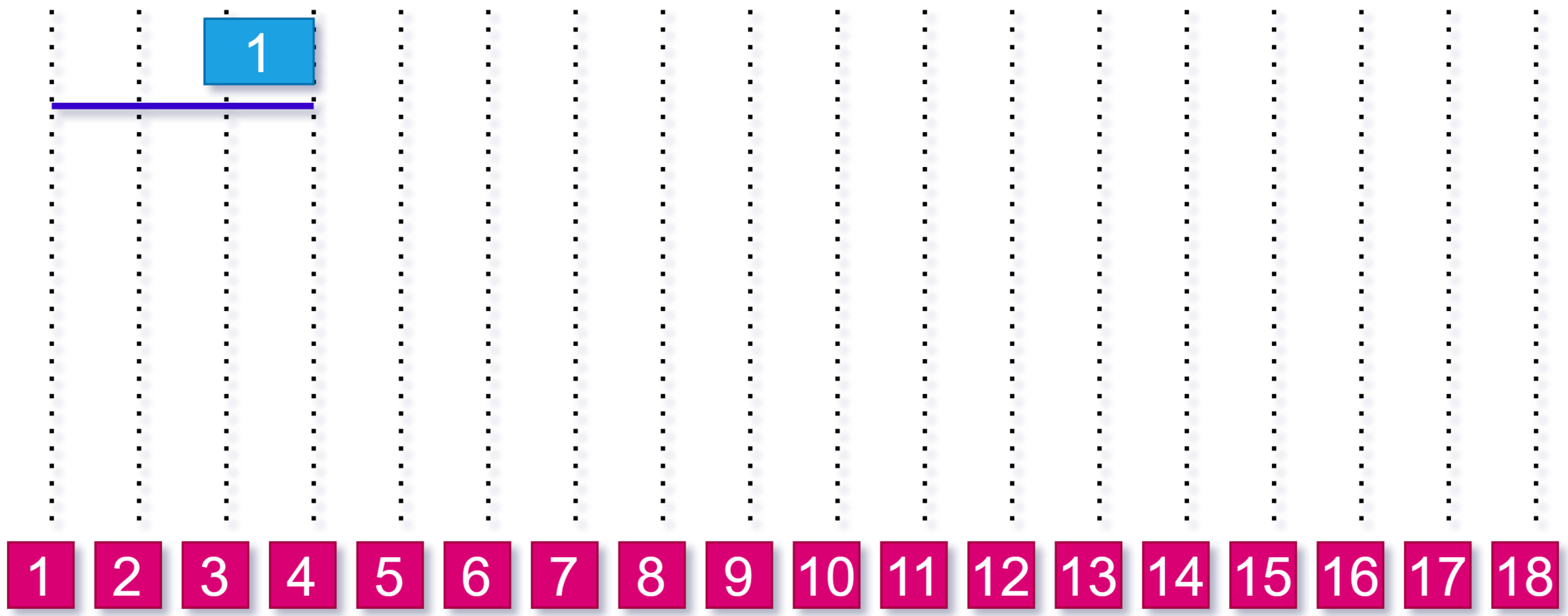
Pseudocode for Greedy Algorithm:

$$
\begin{aligned}
&\mathrm{GAS}(s, f, n)\ \{ \\
&\quad A \leftarrow \{1\} \\
&\quad j \leftarrow 1 \\
&\quad \text{for } i \leftarrow 2 \text{ to } n \text{ do} \\
&\quad\quad \text{if } s_i \geq f_j \text{ then} \\
&\quad\quad\quad A \leftarrow A \cup \{i\} \\
&\quad\quad\quad j \leftarrow i \\
&\quad\quad \text{endif} \\
&\quad \text{endfor} \\
&\quad \}
\end{aligned}
$$

Greedy Algorithm for Activity Selection, An Example (Step-1)

$$f_j = 0$$

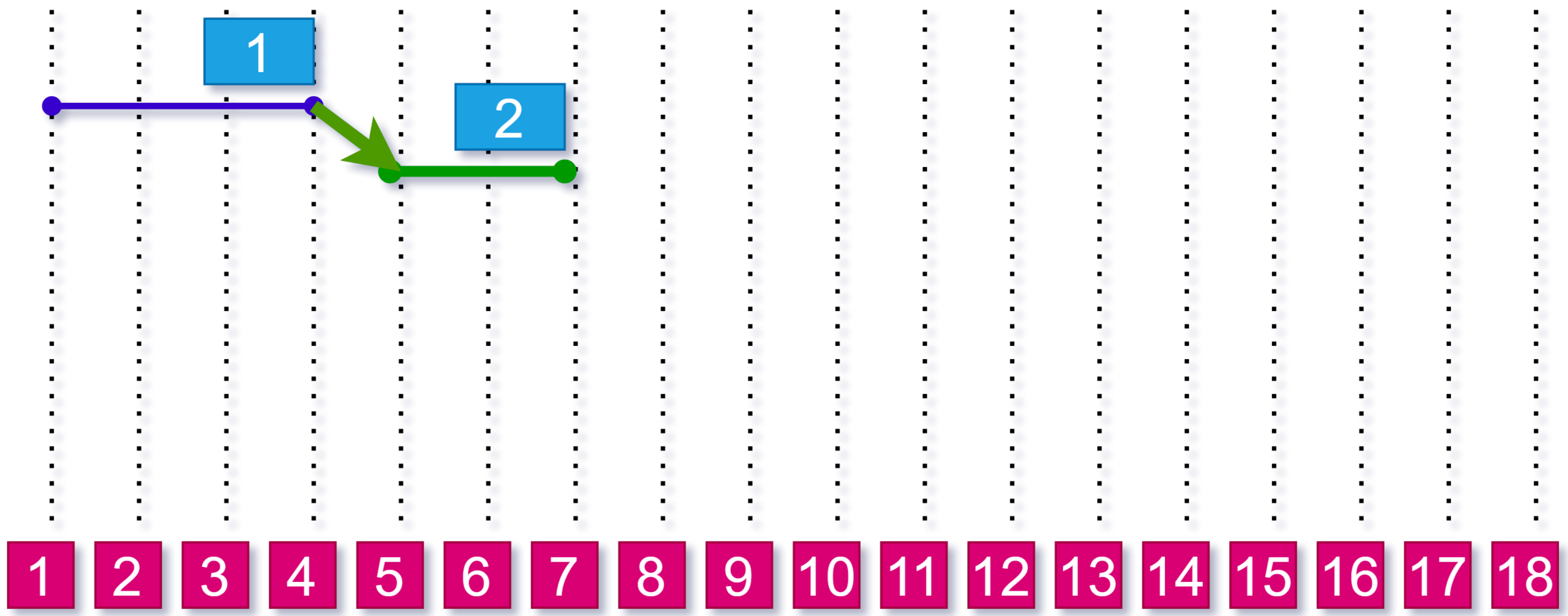
$$S = \{[1,4),[5,7),[2,8),[3,11),[8,15),[13,18)\}$$



Greedy Algorithm for Activity Selection, An Example (Step-2)

$f_j = 4$

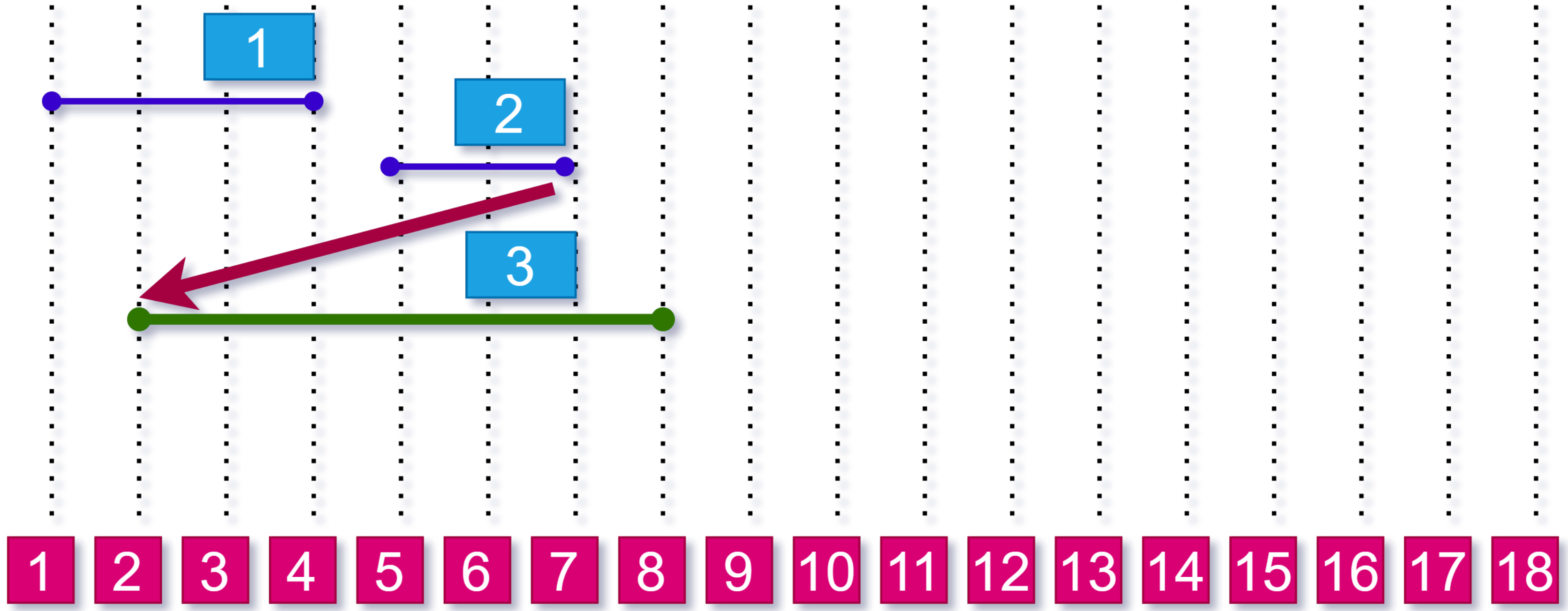
$S = \{[1,4), [5,7), [2,8), [3,11), [8,15), [13,18)\}$



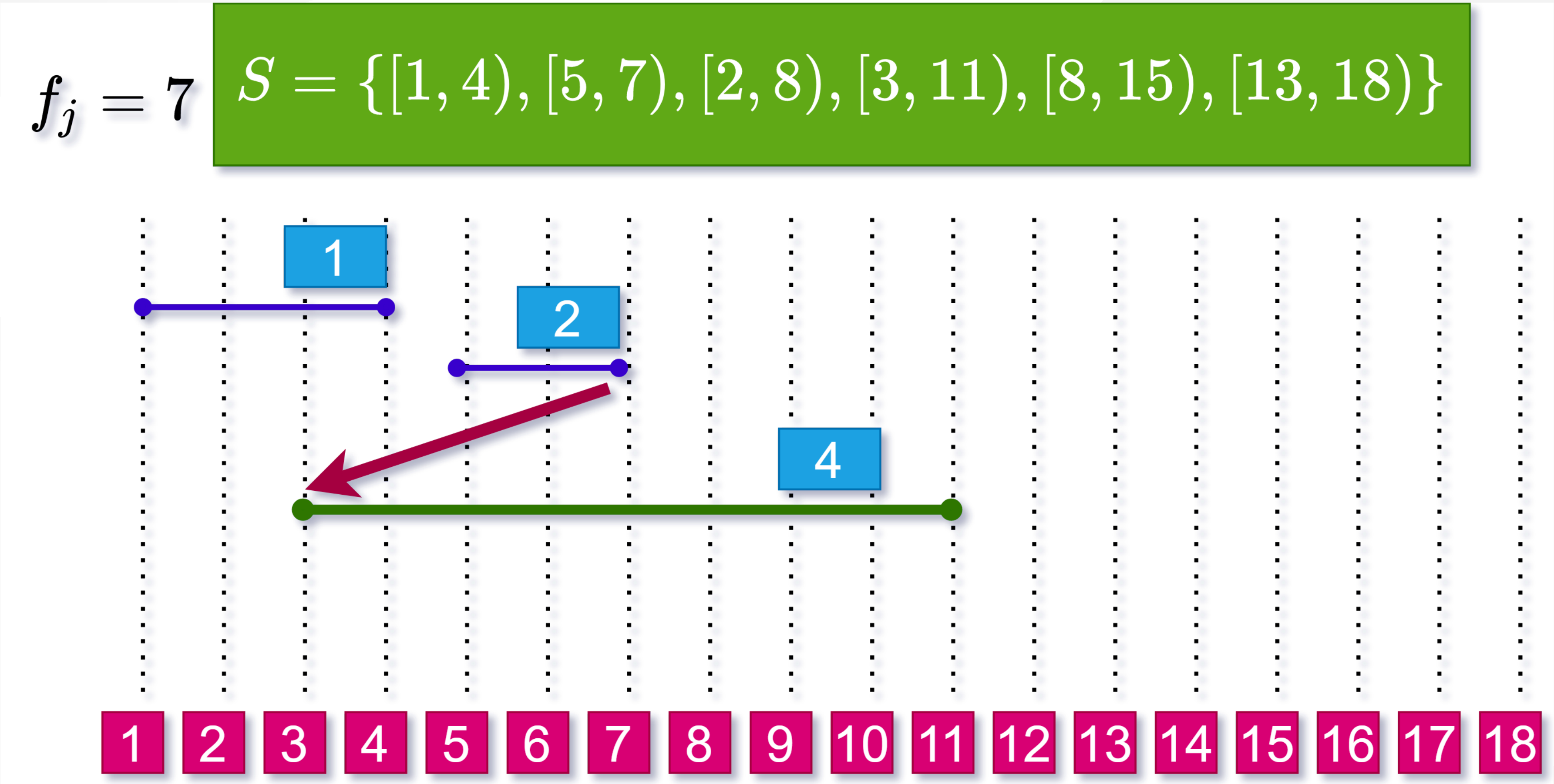
Greedy Algorithm for Activity Selection, An Example (Step-3)

$f_j = 7$

$S = \{[1,4),[5,7),[2,8),[3,11),[8,15),[13,18)\}$



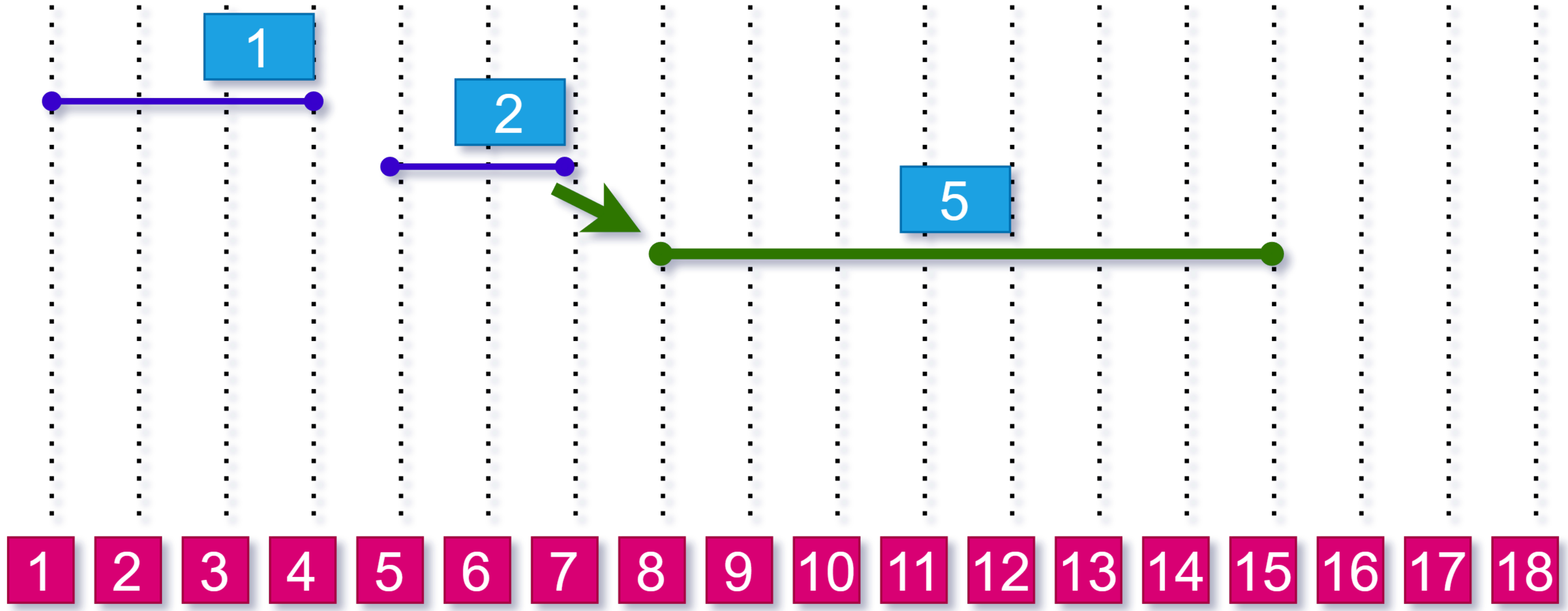
Greedy Algorithm for Activity Selection, An Example (Step-4)



Greedy Algorithm for Activity Selection, An Example (Step-5)

$f_j = 7$

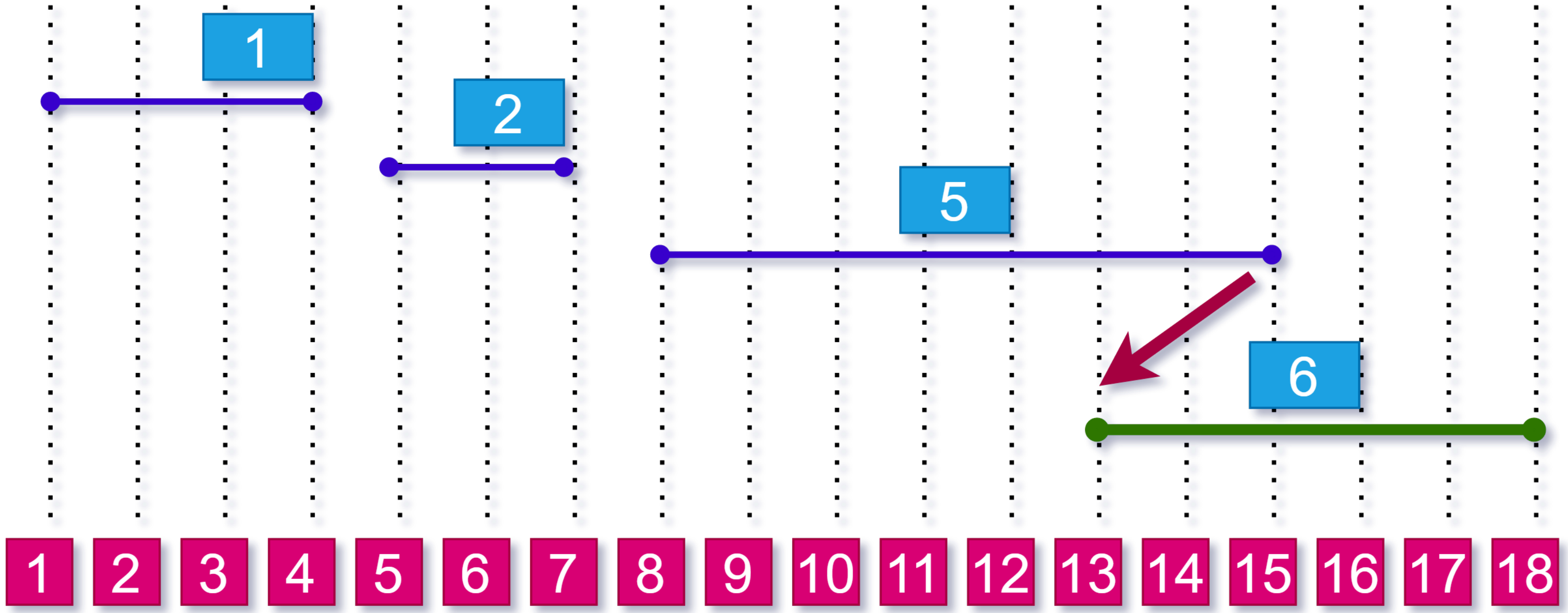
$S = \{[1,4), [5,7), [2,8), [3,11), [8,15), [13,18)\}$



Greedy Algorithm for Activity Selection, An Example (Step-6)

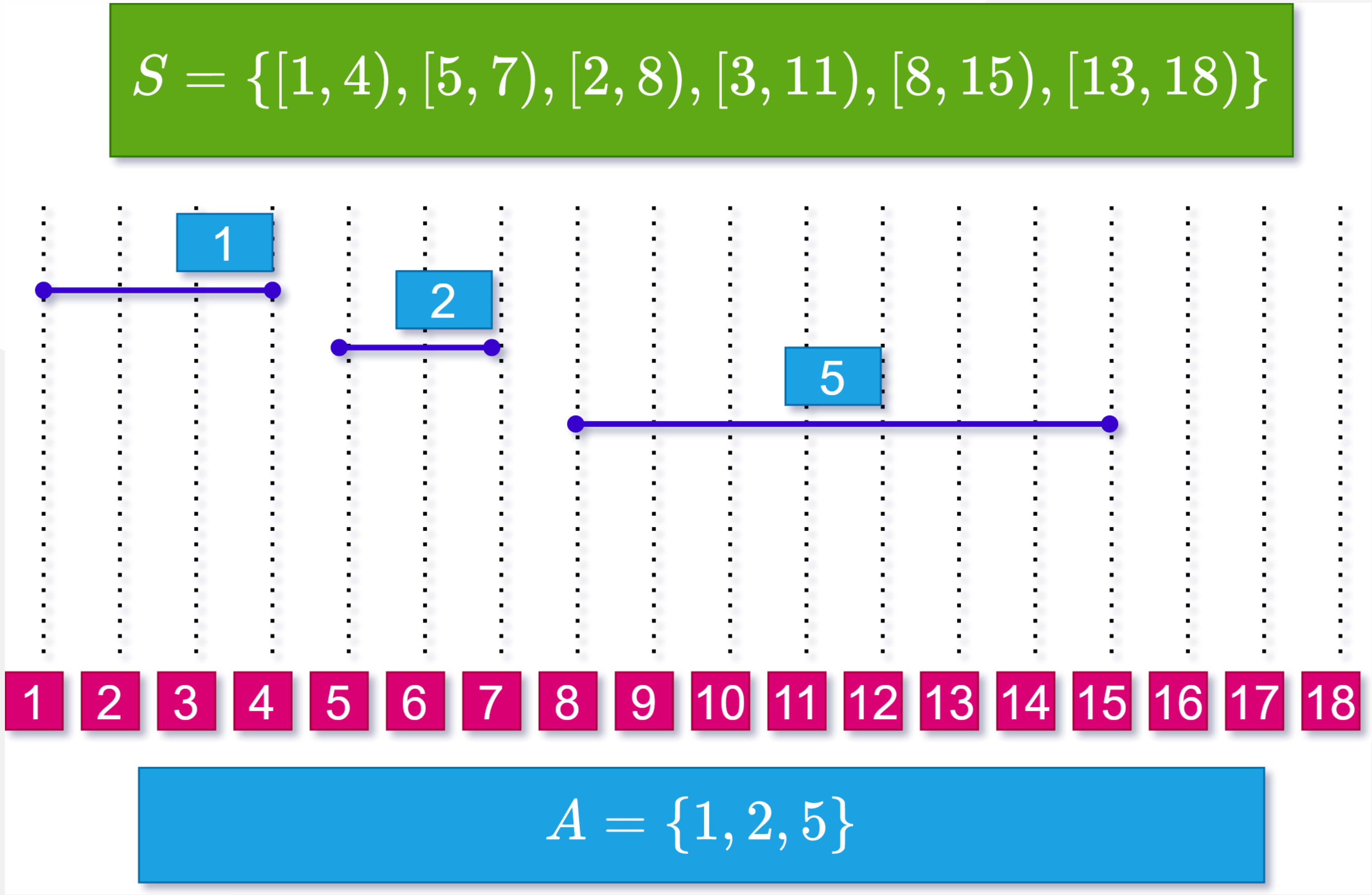
$$f_j = 15$$

$$S = \{[1,4), [5,7), [2,8), [3,11), [8,15), [13,18)\}$$



Greedy Algorithm for Activity Selection, An Example (Step-7)

Final Solution



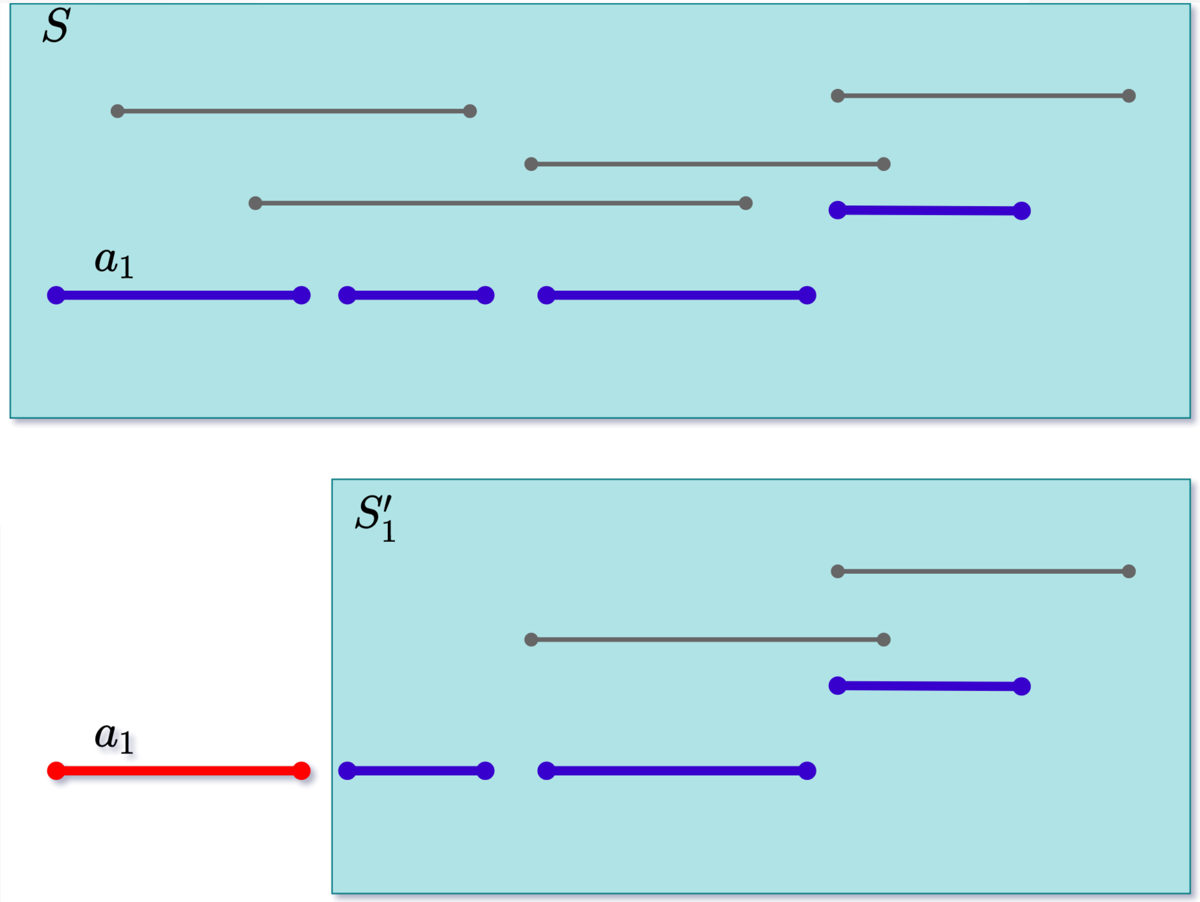
Comparison of DP and Greedy Algorithms

Reminder: DP-Based Matrix Chain Order

$$m_{ij} = \underset{i \leq k < j}{MIN}\{m_{ik} + m_{k+1,j} + p_{i-1}p_kp_j\}$$

- We don't know ahead of time which k value to choose.
- We first need to compute the results of subproblems m_{ik} and $m_{k+1,j}$ before computing m_{ij}
- The selection of k is done based on the **results of the subproblems**.

Greedy Algorithm for Activity Selection



explained in the next slide..

Greedy Algorithm for Activity Selection

- Make a greedy selection in the beginning:
  - Choose a_1 (the activity with the earliest finish time)
- Solve the remaining subproblem S_1' (all activities that start after a1)

Greedy vs Dynamic Programming

- Optimal substructure property exploited by both **Greedy** and **DP** strategies
- **Greedy Choice Property:** A sequence of locally optimal choices $\Rightarrow$ an optimal solution
  - We make the choice that seems best at the moment
  - Then solve the subproblem arising after the choice is made
- **DP:** We also make a choice/decision at each step, but the choice may depend on the optimal solutions to subproblems
- **Greedy:** The choice may depend on the choices made so far, but it cannot depend on any future choices or on the solutions to subproblems

Greedy vs Dynamic Programming

- **DP** is a bottom-up strategy (*use memory to store the results of subproblems*)
- **Greedy** is a top-down strategy (*make choices at each step*)
  - each greedy choice in the sequence iteratively reduces each problem to a similar but smaller problem

Proof of Correctness of Greedy Algorithms

- Examine a globally optimal solution
- Show that this soln can be modified so that
  - (1) A greedy choice is made as the first step
  - (2) This choice reduces the problem to a similar but smaller problem
- Apply induction to show that a greedy choice can be used at every step
- Showing (2) reduces the proof of correctness to proving that the problem exhibits optimal substructure property

Greedy Choice Property - Proof

- **Theorem:** There exists an optimal solution $A \subseteq S$ such that $a_1 \in A$
- **Proof:** Consider an arbitrary optimal solution $B = \{a_k, a_\ell, a_m, \dots\}$, where $f_k < f_\ell < f_m < \dots$
    - If $k = 1$, then B starts with a_1, and the proof is complete
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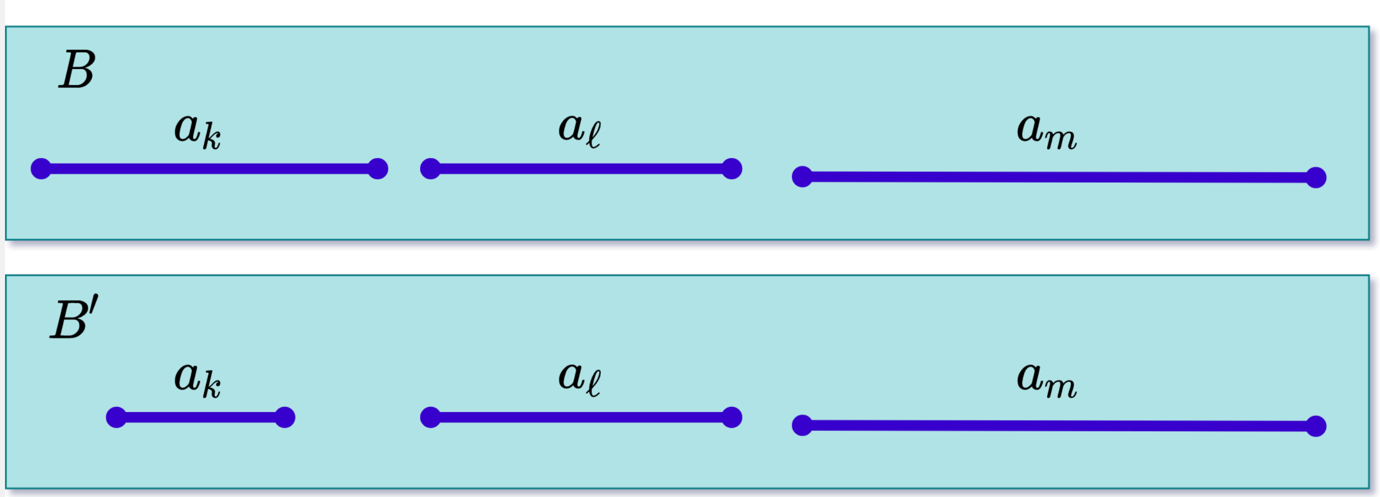


Elements of Greedy Strategy

- How can you judge whether
- A greedy algorithm will solve a particular optimization problem?
- **Two key ingredients**
  - Greedy choice property
  - Optimal substructure property

Key Ingredients of Greedy Strategy

- **Greedy Choice Property:** A globally optimal solution can be arrived at by making locally optimal (greedy) choices
- In **DP**,we make a choice at each step but the choice may depend on the solutions to subproblems
- In **Greedy Algorithms**, we make the choice that seems best at that moment then solve the subproblems arising after the choice is made
  - The choice may depend on choices so far, but it cannot depend on any future choice or on the solutions to subproblems
- DP solves the problem bottom-up
- Greedy usually progresses in a top-down fashion by making one greedy choice after another reducing each given problem instance to a smaller one

Key Ingredients: Greedy Choice Property

- We must prove that a greedy choice at each step yields a globally optimal solution
- The proof examines a globally optimal solution
- Shows that the soln can be modified so that a **greedy choice made as the first step** reduces the problem to a similar but smaller subproblem
- Then **induction** is applied to show that a greedy choice can be used at each step
- Hence, this induction proof reduces the proof of correctness to demonstrating that an optimal solution must exhibit **optimal substructure** property

Key Ingredients: Greedy Choice Property

- **How to prove the greedy choice property?**
  - Consider the greedy choice c
  - Assume that there is an **optimal solution** B **that doesn't contain** c.
  - Show that it is possible to **convert** B to another optimal solution B', where B' contains c.
- **Example:** Activity selection algorithm
  - Greedy choice: a_1 (the activity with the earliest finish time)
  - Consider an optimal solution B without a_1
  - Replace the first activity in B with a_1 to construct B'
  - Prove that B' must be an optimal solution

Key Ingredients: Optimal Substructure

- A problem exhibits optimal substructure if an optimal solution to the problem contains within it optimal solutions to subproblems
- **Example:** Activity selection problem S
  - If an optimal solution A to S begins with activity a1 then the set of activities

$$A' = A - \{a_1\}$$

  - is an optimal solution to the activity selection problem

$$S' = \{a_i \in S : s_i \geq f_1\}$$

  - where s_i is the start time of activity a_i and f_i is the finish time of activity a_i

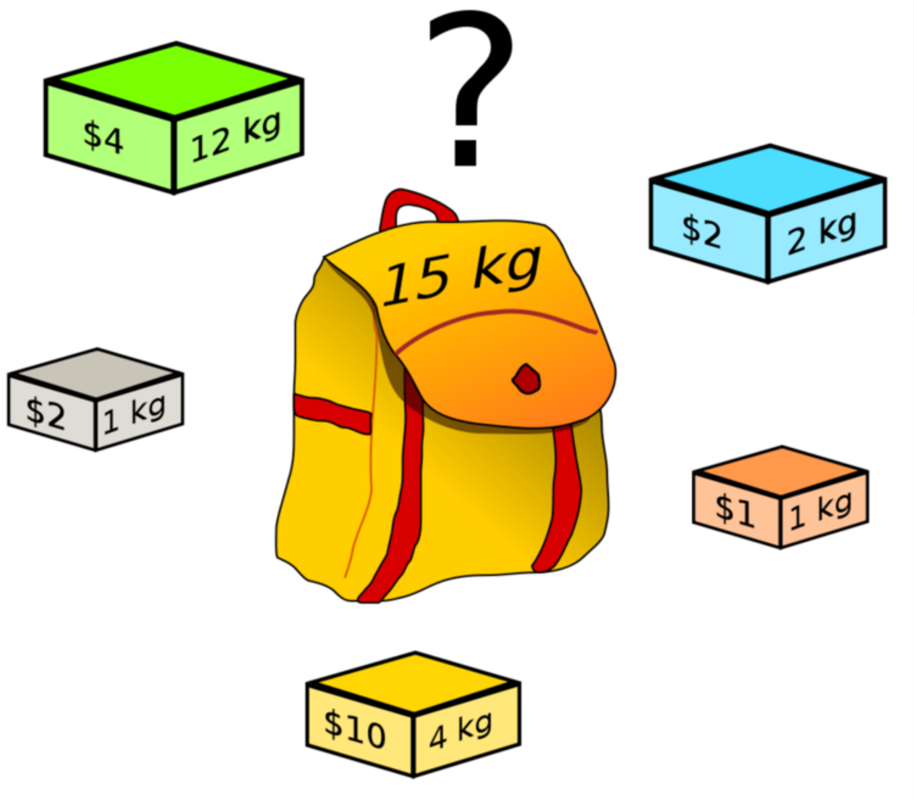
Key Ingredients: Optimal Substructure

- Optimal substructure property is exploited by both Greedy and dynamic programming strategies
- Hence one may
  - Try to generate a dynamic programming soln to a problem when a greedy strategy suffices inefficient
  - Or, may mistakenly think that a greedy soln works when in fact a DP soln is required incorrect
- **Example:** `Knapsack Problems(S, w)`

Knapsack Problems

Knapsack Problem

- Each item i has:
  - weight w_i
  - value v_i
- A thief has a knapsack of weight capacity w
- **Which items to choose to maximize the value of the items in the knapsack?**



Knapsack Problem: Two Versions

- **The 0-1 knapsack problem:**
  - Each item is discrete.
  - Each item either chosen as a whole or not chosen.
  - **Examples:** *TV, laptop, gold bricks, etc.*
- **The fractional knapsack problem:**
  - Can choose fractional part of each item.
  - If item i has weight wi, we can choose any amount ≤ wi
  - **Examples:** *Gold dust, silver dust, rice, etc.*

Knapsack Problems

- **The 0-1 Knapsack Problem(S, W)**
  - A thief robbing a store finds n items $S = \{I_1, I_2, \dots, I_n\}$, the ith item is worth v_i dollars and weighs w_i pounds, where vi and wi are integers
  - He wants to take as valuable a load as possible, but he can carry at most W pounds in his knapsack, where W is an integer
  - The thief cannot take a fractional amount of an item
- **The Fractional Knapsack Problem (S, W)**
  - The scenario is the same
  - But, the thief can take fractions of items rather than having to make binary ($0-1$) choice for each item

Optimal Substructure Property for the 0-1 Knapsack Problem (S, W)

- Consider an optimal load L for the problem (S, W).
- Let Ij be an item chosen in L with weight wj
- Assume we remove Ij from L, and let:

$$L'_j = L - \{I_j\}$$
$$S'_j = S - \{I_j\}$$
$$W'_j = W - w_j$$

- Q: *What can we say about the optimal substructure property?*

Optimal Substructure Property for the 0-1 Knapsack Problem (S, W)

$$L'_j = L - \{I_j\}$$
$$S'_j = S - \{I_j\}$$
$$W'_j = W - w_j$$

- Optimal substructure property:
  - L'_j must be an optimal solution for (S'_j, W'_j)
- **Why?**
  - If we remove item j from L, we can construct a new optimal solution L'_j for (S'_j, W'_j)
  - If L'_j is optimal, then L must be optimal

Optimal Substructure Property for the 0-1 Knapsack Problem (S, W)

$$L_j' = L – \{I_j\}$$
$$S_j' = S – \{I_j\}$$
$$W_j' = W – w_j$$

- Optimal substructure: L_j' must be an optimal solution for (S_j', W_j')
- **Proof:** By contradiction, assume there is a solution B_j' for (S_j', W_j'), which is better than L_j'.
  - We can construct a solution B for the original problem (S, W) as: $B = Bj' \cup Ij$.
  - The total value of B is now higher than L, which is a contradiction because L is optimal for (S, W).
- $Q.E.D.$

Optimal Substructure Property for the Fractional Knapsack Problem (S, W)

- Consider an optimal solution L for (S, W)
- If we remove a weight $0 < w \leq w_j$ of item j from optimal load L and let:
  - The remaining load

$$L_j' = L - \{w \text{ pounds of } I_j\}$$

  - must be a most valuable load weighing at most

$$W_j' = W - w$$

  - pounds that the thief can take from

$$S_j' = S - \{I_j\} \cup \{w_j - w \text{ pounds of } I_j\}$$

- That is, Lj´ should be an optimal soln to the

$$\text{Fractional Knapsack Problem}(S_j', W_j')$$

Knapsack Problems

- Two different problems:
  - 0-1 knapsack problem
  - Fractional knapsack problem
- The problems are similar.
  - Both problems have optimal substructure property.
- Which algorithm to solve each problem?

Fractional Knapsack Problem

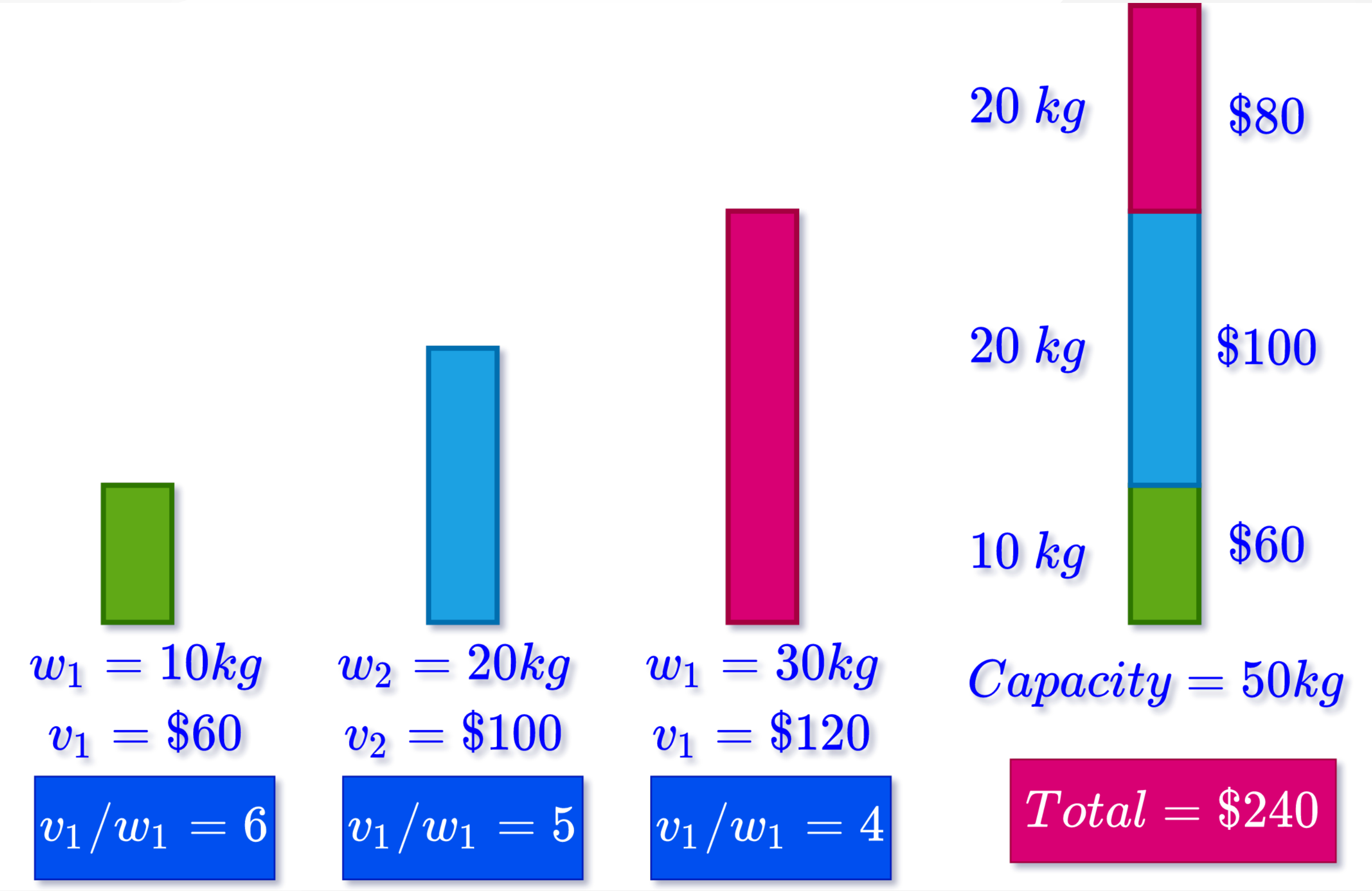
- Can we use a greedy algorithm?
- Greedy choice: Take as much as possible from the item with the largest value per pound v_i/w_i
- Does the greedy choice property hold?
  - Let j be the item with the largest value per pound v_j/w_j
  - Need to prove that there is an optimal load that has as much j as possible.
  - **Proof:** *Consider an optimal solution L that does not have the maximum amount of item j. We could replace the items in L with item j until L has maximum amount of j. L would still be optimal, because item j has the highest value per pound.*

Greedy Solution to Fractional Knapsack

- (1) Compute the value per pound v_i/w_i for each item
- (2) The thief begins by taking, as much as possible, of the item with the greatest value per pound
- (3) If the supply of that item is exhausted before filling the knapsack, then he takes, as much as possible, of the item with the next greatest value per pound
- (4) Repeat (2-3) until his knapsack becomes full

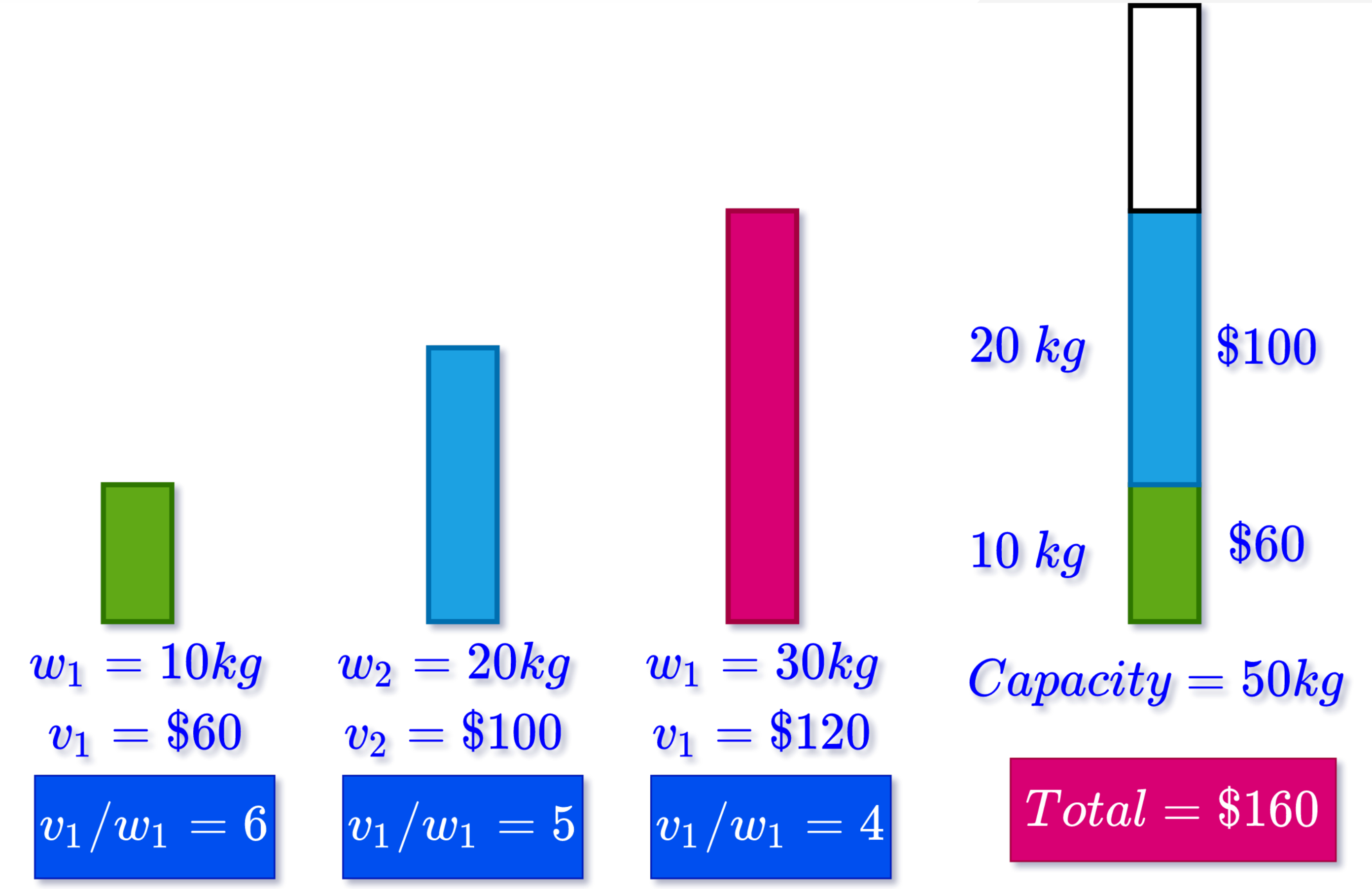
Thus, by sorting the items by value per pound the greedy algorithm runs in $O(nlgn)$ time

Fractional Knapsack Problem: Example



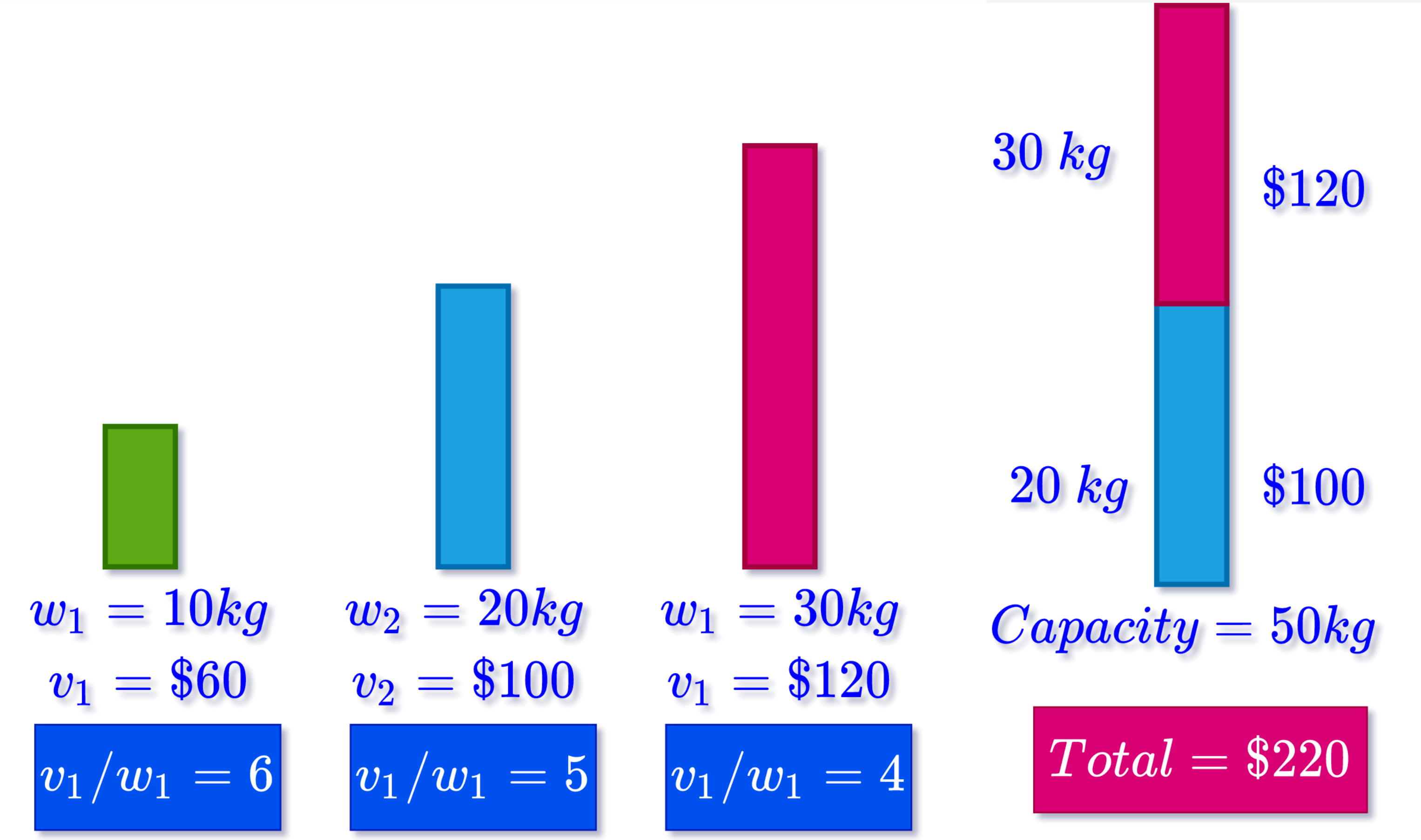
0-1 Knapsack Problem

- Can we use the same greedy algorithm?
  - Is there a better solution?



0-1 Knapsack Problem

- The optimal solution for this problem is:
  - This solution cannot be obtained using the greedy algorithm



0-1 Knapsack Problem

- When we consider an item I_j for inclusion we must compare the solutions to two subproblems
  - Subproblems in which I_j is included and excluded
- The problem formulated in this way gives rise to many
  - **overlapping subproblems** (a key ingredient of DP)
    - In fact, dynamic programming can be used to solve the **0-1 Knapsack problem**

0-1 Knapsack Problem

- A thief robbing a store containing n articles
  - $\{a_1, a_2, \dots, a_n\}$
- The value of i_{th} article is v_i dollars (v_i is integer)
- The weight of i_{th} article is w_i kg (w_i is integer)
- Thief can carry at most W kg in his knapsack
- Which articles should he take to maximize the value of his load?
- Let $K_{n,W} = \{a_1, a_2, \dots, a_n : W\}$ denote 0-1 knapsack problem
- Consider the solution as a sequence of n decisions
  - **i.e.**, i_{th} decision: whether thief should pick a_i for optimal load.

Optimal Substructure Property

- Notation: $K_{n,W}$:
  - The items to choose from: $\{a_1, \ldots, a_n\}$
  - The knapsack capacity: W
- Consider an optimal load L for problem $K_{n,W}$
- Let's consider two cases:
  - a_n is **in** L
  - a_n is **not in** L

Optimal Substructure Property

- **Case 1:** If $a_n \in L$
  - What can we say about the optimal substructure?
    - $L – \{a_n\}$ must be optimal for $K_{n-1,W-wn}$
    - $K_{n-1,W-wn}$:
      - The items to choose from $\{a_1, \ldots a_{n-1}\}$
      - The knapsack capacity: $W – wn$
- **Case 2:** If $a_n \notin L$
  - What can we say about the optimal substructure?
    - L must be optimal for $K_{n-1,W}$
      - $K_{n-1,W}$:
      - The items to choose from $\{a_1, \ldots a_{n-1}\}$
      - The knapsack capacity: W

Optimal Substructure Property

- In other words, optimal solution to $K_{n,W}$ contains an optimal solution to:
  - either: $K_{n-1,W-wn}$ (if a_n is selected)
  - or: $K_{n-1,W}$ (if a_n is not selected)

Recursive Formulation

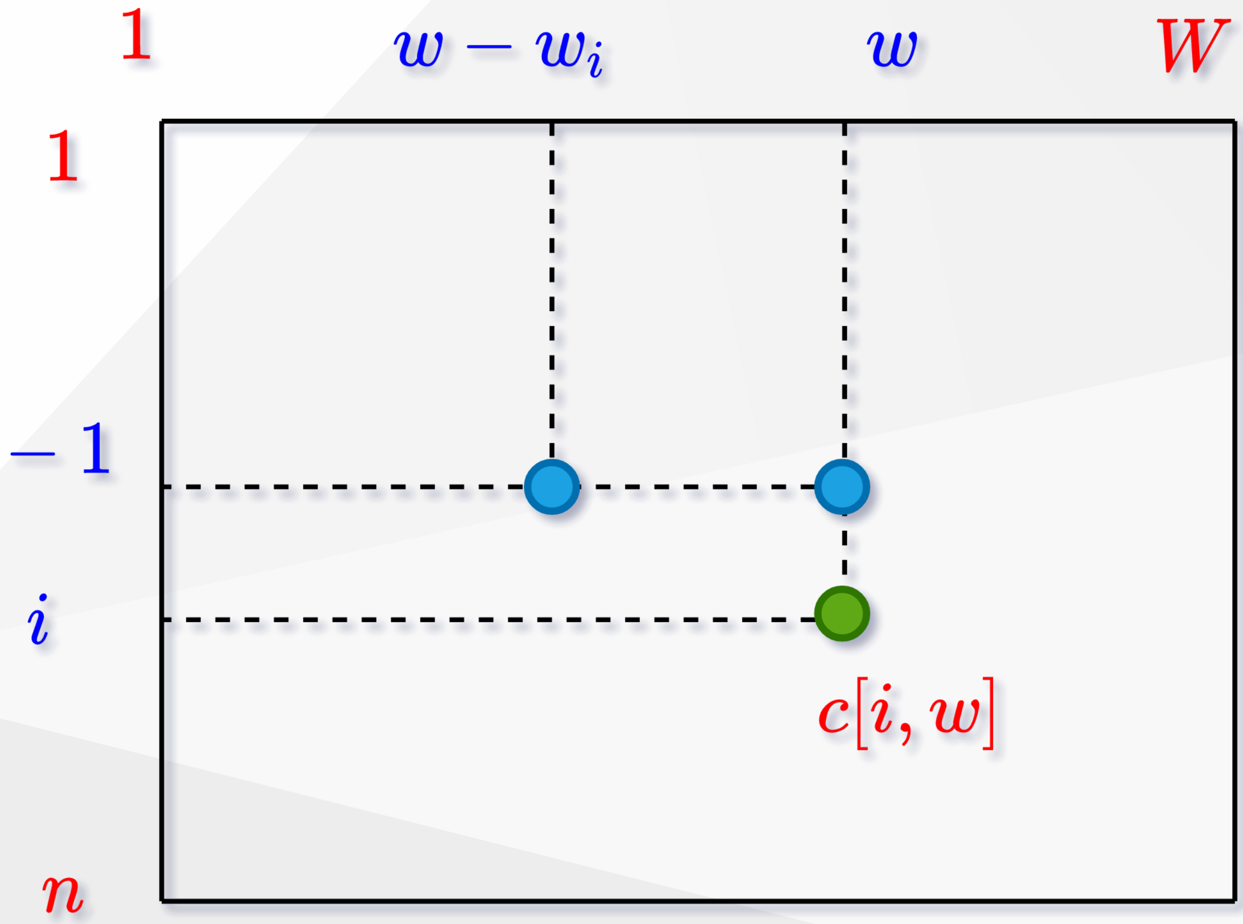
$$c[i,w] = \begin{cases} 0 & \text{if } i = 0, \text{ or } w = 0 \\ c[i-1,w], & \text{if } w_i > w \\ max\{v_i + c[i-1, w-w_i], c[i-1,w]\} & otherwise \end{cases}$$

0-1 Knapsack Problem

- **Recursive definition for value of optimal soln:**
  - This recurrence says that an optimal solution $S_{i,w}$ for $K_{i,w}$
    - either contains $a_i \Rightarrow c(S_i, w) = v_i + c(S_{i-1,w-w_i})$
    - or does not contain $a_i \Rightarrow c(Si, w) = c(S_{i-1}, w)$
  - If thief decides to pick a_i
    - He takes v_i value and he can choose from $\{a_1, a_2, \dots, a_{i-1}\}$ up to the weight limit $w - w_i$ to get $c[i-1, w-w_i]$
  - If he decides not to pick a_i
    - He can choose from $\{a_1, a_2, \dots, a_{i-1}\}$ up to the weight limit w to get $c[i-1, w]$
  - The better of these two choices should be made

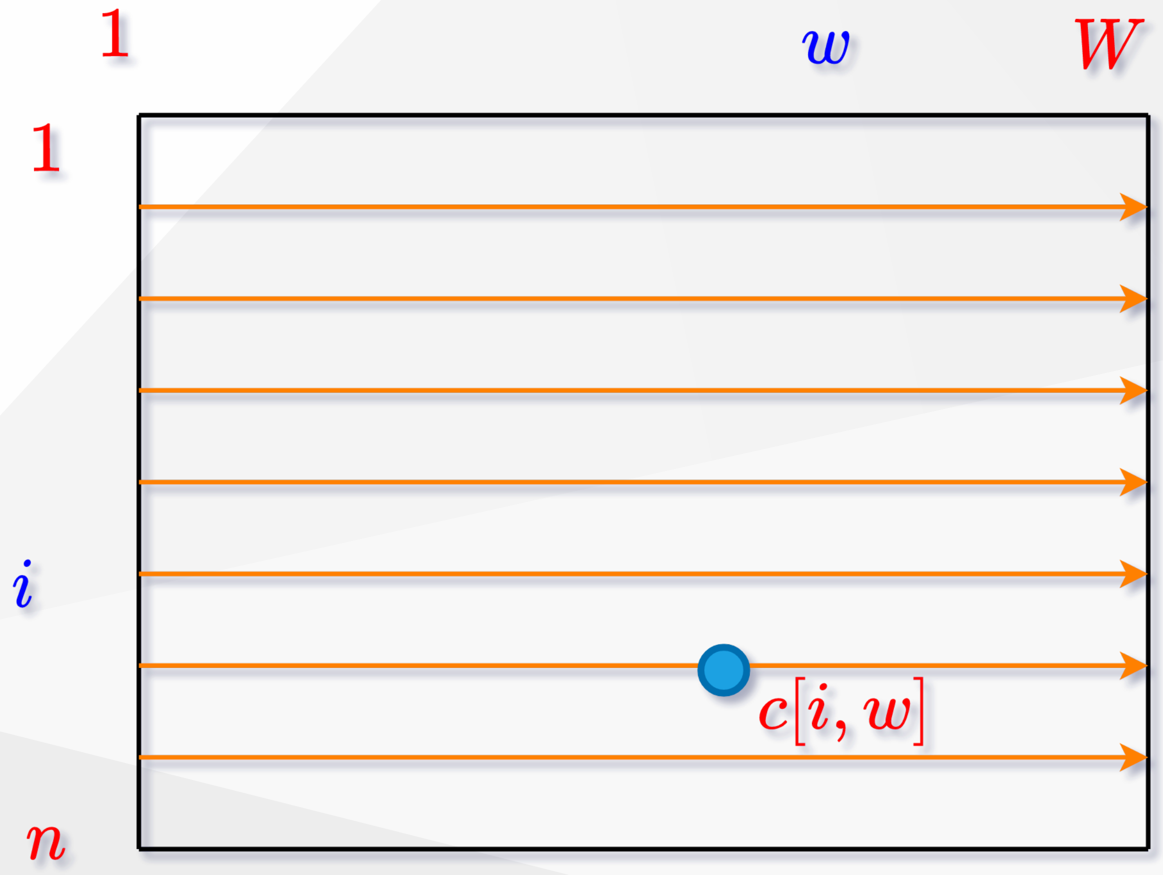
Bottom-up Computation

- Need to process:
  - $c[i, w]$
- after computing:
  - $c[i-1, w]$,
  - $c[i-1, w-w_i]$
    - for all $w_i < w$



Bottom-up Computation

$$
\begin{array}{l}
\text{for } i \leftarrow 1 \text{ to } n \text{ do} \\
\quad \text{for } w \leftarrow 1 \text{ to } W \text{ do} \\
\qquad \ldots \\
\qquad c[i, w] \leftarrow \ldots \\
\qquad \ldots
\end{array}
$$



DP Solution to 0-1 Knapsack

- c is an $(n+1) \times (W+1)$ array; $c[0 \dots n : 0 \dots W]$
- **Note** : table is computed in row-major order
- **Run time:** $T(n) = \Theta(nW)$

DP Solution to 0-1 Knapsack

$$
\begin{aligned}
&\text{KNAP0-1}(v, w, n, W)\\
&\quad \text{for } \omega \leftarrow 0 \text{ to } W \text{ do}\\
&\quad\quad c[0, \omega] \leftarrow 0\\
&\quad \text{for } i \leftarrow 0 \text{ to } m \text{ do}\\
&\quad\quad c[i, 0] \leftarrow 0\\
&\quad \text{for } i \leftarrow 0 \text{ to } m \text{ do}\\
&\quad\quad\quad \text{for } \omega \leftarrow 1 \text{ to } W \text{ do}\\
&\quad\quad\quad\quad \text{if } w_i \leq \omega \text{ then}\\
&\quad\quad\quad\quad\quad c[i, \omega] \leftarrow max\{v_i + c[i-1, \omega - w_i], c[i-1, \omega]\}\\
&\quad\quad\quad\quad \text{else}\\
&\quad\quad\quad\quad\quad c[i, \omega] \leftarrow c[i-1, \omega]\\
&\quad return\ c[m, W]
\end{aligned}
$$

Constructing an Optimal Solution

- No extra data structure is maintained to keep track of the choices made to compute $c[i, w]$
  - i.e. The choice of whether choosing item i or not
- Possible to understand the choice done by comparing $c[i, w]$ with $c[i-1, w]$
  - If $c[i, w] = c[i-1, w]$ then it means item i was assumed to be not chosen for the best $c[i, w]$

Finding the Set S of Articles in an Optimal Load

$$\text{SOLKNAP0-1}(a, v, w, n, W, c)$$

$$i \leftarrow n; \omega \leftarrow W$$

$$S \leftarrow \emptyset$$

$$while\; i \leftarrow 0\; do$$

$$if\; c[i, \omega] = c[i-1, \omega]\; then$$

$$i \leftarrow i - 1$$

$$else$$

$$S \leftarrow S \cup \{a_i\}$$

$$\omega \leftarrow \omega - w_i$$

$$i \leftarrow i - 1$$

$$return\; S$$

References

- Introduction to Algorithms, Third Edition | The MIT Press
- Bilkent CS473 Course Notes (new)
- Bilkent CS473 Course Notes (old)

$$-End-Of-Week-7-Course-Module-$$